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ABSTRACT

Chapters 11 through 14 of a 14-chapter, three-volume work on arithmetical applications are contained in this document. Each chapter details the "use classes" of one broad arithmetical concept. (A "use class" of a concept is a set of examples of real world uses of the concept which share a common structure). Each chapter contains: an introduction and summary; four sections, each devoted to one use class and containing a general introduction, questions, and comments; suggestions for teaching or illustrating a given concept; questions which test understanding of the ideas presented; and notes and commentary, with reasons for selecting particular use classes, related research, and short essays on issues related to applying the concepts. Topics of the chapters focus on some aspect of "maneuvers," a term coined to group together four types of processes: rewriting, for example, from fractions to decimals (chapter 11); estimating, for example, 3.14 for pi (chapter 12); transforming, for example, standard scores from raw scores (chapter 13); and displaying, for example, graphical data (chapter 14). Each chapter in this part is divided into four sections which focus on, respectively, constraints, clarity, facility, and consistency. A five-page bibliography and index for topics in the three volumes are included. (JN)

APPLYING ARITHMETIC

A HANDBOOK OF APPLICATIONS OF ARITHMETIC

PART III: MANEUVERS

ARITHMETIC AND ITS APPLICATIONS PROJECT
DEPARTMENT OF EDUCATION
THE UNIVERSITY OF CHICAGO

APPLYING ARITHMETIC

A HANDBOOK OF APPLICATIONS OF ARITHMETIC

PART III: MANEUVERS

by

ZALMAN USISKIN AND MAX BELL

under the auspices of the

ARITHMETIC AND ITS APPLICATIONS PROJECT

DEPARTMENT OF EDUCATION

THE UNIVERSITY OF CHICAGO

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Preface

The writing of this volume has been motivated by two existing gaps in mathematics education. The first gap is between student performance on arithmetic skills and the generally worse performance on realistic problems utilizing those same skills. The second gap is the disparity between oft-stated goals of professional organizations and schools and textbooks (generally supportive of applications of arithmetic) and the classroom reality. After grade 4, realistic applications of arithmetic do not often appear in the classroom, and those that do appear represent only a narrow picture of this broad domain.

The intended audiences are diverse. First, we have designed the book for use by teachers. Each concept is illustrated by a large number of examples, and comments are given following the examples to aid in adaptation for use in classrooms. Each chapter contains a special section entitled "Pedagogical Remarks" to further assist in this task.

Second, we have designed the book for use by those interested in curriculum design or research. Each chapter contains an extended discussion of selected theoretical, pedagogical, philosophical, psychological or semantic issues and research related to the ideas found within the chapter.

Third, because teachers and other professional educators often encounter books like this one only in the context of coursework, we have included a number of questions at the end of each chapter.

Fourth, we hope that the ideas in this book might also be suitable to lay readers interested in understanding the uses of arithmetic. We have tried to make the writing easy to understand and in most places the mathematical prerequisites necessary to comprehend the material are minimal.

Our goal is to improve our society's understanding of the applications of arithmetic. In the past, due to the necessity of having to spend a great deal of time teaching how to get answers, books could not afford to be devoted to teaching when to use particular arithmetic processes. Calculators, in our opinion, allow us to change emphasis from how to when. This book constitutes a first attempt to provide a rather complete categorization of the simpler applications of arithmetic.

The organization of this book is not definitive and in many places may not exhaust the range of applications. Many may disagree with our categorizations. We encourage criticism; we only hope that those who criticize will help us improve the ideas presented here or produce their own improved version.

Zalman Usiskin and Max Bell
June, 1983

Acknowledgements

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The Arithmetic and Its Applications project was assisted in its work by two advisory boards, one consisting of university personnel, the other of junior high school or middle school teachers and supervisors. The advisory board members were: Pamela Ames, Harry Bohan, Sherye Garmony, Alan Hoffer, Jeremy Kilpatrick, James McBride, Kay Nebel, and Jane Swafford. Roberta Dees worked with us on this project for a year. Each of these people assisted in the development of this manuscript in his or her own way (but the authors take full responsibility for the writing).

Early drafts of these materials were tried out in classes at the University of Chicago by us, Sam Houston State University by Harry Bohan, and Ohio State University by Alan Hoffer. We appreciate the willingness of these institutions to support this endeavor and extend our thanks to the students who gave comments to help us improve it.

Our thanks go also to the University of Chicago for providing facilities, colleagues, and students particularly amenable to the kind of thinking this type of writing requires.

Finally, we are each fortunate to have wives who are not only supportive of our work but who also are involved in mathematics

education. They have been responsive sounding boards for most of the ideas presented here and often were the ones who provided an ultimate clarification of an issue. We appreciate their help more than we can put in words.

Zalman Usiskin and Max Bell

Introduction

For most of us, an application of arithmetic begins with an attempt to comprehend numbers we encounter in everyday living. These range from prices of goods and services to interest rates on investments to sports scores to minimum daily nutrient requirements to ID numbers to geographic information found on roads and maps to technical information about objects around the home to results of surveys published in newspapers or magazines. Our society has become increasingly numericized, requiring each of us to process more numbers than many of us thought we would need.

On many occasions, comprehension of numerical information suffices. We only may want to know the protein content of a food, or a sports score, or the time to the airport, or a social security number. At other times we may wish to operate on given numerical information to generate more information. From prices of foods, one may calculate which is more economical and still supply nutritional needs. From interest rates, income can be determined. From temperature data, energy costs can be estimated. From sports data, decisions regarding the quality of teams and participants may be desired. From information about the size of living quarters, wall and floor covering needs can be established. We add, subtract, multiply, divide, take powers, and apply other operations of arithmetic to help us obtain the additional numerical information.

But things are not always so simple. Given numerical information is not always written in a form that makes it easy to operate upon.

We do not always know what to do with such information until we display, scale, or estimate it in some way. We classify rewriting, graphing, scaling, and estimating as maneuvers and recognize that we often maneuver both given numerical information and the results of computations.

These three application skills comprise the subject matter of this book, one part of the book being devoted to each of them.

In	Part I,	Numbers	we ask	To what uses are numbers and number aggregates put?
In	Part II,	Operations	we ask	What are the common uses of the fundamental operations?
In	Part III,	Maneuvers	we ask	For what reasons are the most common types of maneuvers applied

The three parts are divided into a total of 14 chapters. Each chapter details the use classes of one broad arithmetic concept (e.g., single number, multiplication, or estimation). The notion of use class is at the heart of this book and is roughly defined here.

> A use class of a concept is a set of examples of real world uses of the concept which share a common structure.

The arithmetic concepts in this book have from 3 to 6 use classes each; there are 57 use classes in the 14 chapters. The chapters are organized in the following way.

Introduction

3-6 Sections, one devoted to each use class, each with a general introduction to that class followed by example questions with answers and comments

Summary

Pedagogical Remarks

Questions

Notes and Commentary

Since use classes are defined in terms of examples, the major space in this volume is devoted to the examples, answers, and comments. The purpose of the other components of each chapter is as follows:

The introduction and summary each contain a short synopsis of the types of applications of a particular concept.

Suggestions for teaching or illustrating a given concept may be found both in the comments following each example and in the pedagogical remarks.

The questions are a test of the reader's understanding of the ideas herein. The notes and commentary include our reasons for the selection of the particular use classes, related research, and short essays on issues related to applying the various concepts.

A calculator is strongly recommended for all sections of this book so that the reader can spend time dealing with the concepts of this book rather than with paper and pencil computation. A calculator with an x^y key is necessary in Chapters 9 and 10.

APPLYING ARITHMETIC

Table of Contents

Part I: Numbers (continued)

PART III: MANEUVERS

(Printed in the Chicago Tribune, 12/10/80)

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Introduction

The steps in the process of applying arithmetic are rather straightforward and simple. One confronts or generates numbers in a real world situation, decides what operations (if any) are appropriate, performs the operations, translates the results back into the original situation, and checks the feasibility of the results in that situation. Parts I and II of this book have detailed the ways in which number objects are used and the situations in which the most common operations are performed. Except for checking feasibility, a task which depends upon the situation (and not the mathematics), since textbooks and calculators can handle performance of operations, it would seem that the first two parts have covered all of the basics in applying arithmetic. To some extent this is true, but there is one general consideration that the steps ignore: numerical information is not always presented in ways clear enough or suitable for future steps in the application process. When this occurs, maneuvers must be performed.

The term "maneuver" as used here, has been coined by us to group together four types of processes.

Rewriting (e.g., from fractions to decimals)

Estimating (e.g., 3.14 for π)

Transforming (e.g., standard scores from raw scores)

Displaying (e.g., graphing data)

The Doonesbury cartoon on the previous page displays a quite unusual maneuver somewhere between estimating and transforming.

Maneuvers are performed to offer the user new ways of looking at existing information and provide ways of obtaining information that may not have been previously accessible. They differ from the arithmetic operations discussed in Part II in that the results of maneuvers do not yield counts, measures, locations, or comparisons of different things, but rather provide a different way of looking at an old thing.

To categorize the uses of the various types of maneuvers, we ask the same sort of question asked in the earlier parts, namely: For what reasons is each used? The answers to this question are broadly the same for each type of maneuver, and each chapter discusses the same four use classes. However, the uses interact with the maneuvers in different ways.

CHAPTER 11

REASONS FOR REWRITING

The same number can be denoted in many ways, as illustrated for the number eight below.

8	16/2	800%	2**3	4 + 4
$512^{1/3}$	$\frac{64}{8}$	VIII	2^3	$.008 \times 10^3$
𝍸 |||	1000_2	22_3	$2\uparrow 3$	$\sqrt{64}$

Just as numbers can be written in many ways, so can quantities, as illustrated for the quantity one yard.

36 in.	3 ft	$\frac{1}{1760}$ mi
91.44 cm	.9144 m	9144 mm
2 ft, 12 in.	2 ft + 1 ft	5 hr × .2 $\frac{\text{yd}}{\text{hr}}$

In using numbers or quantities, we choose the form we desire from the many possibilities. We call changing from one of these forms to another by the global term rewriting. Some books call this renaming. Rewriting or renaming includes a variety of arithmetic activities. Some of the more common are listed here.

reducing (simplifying) fractions

changing fractions to decimals

evaluating numerical expressions

converting from one unit to another

writing as a percent

putting numbers into scientific notation

There are many reasons for rewriting. These reasons can broadly be described by the titles of the four use classes for this chapter:

A. Constraints (e.g., changing 2^3 to 2**3 for use in some computer languages)

B. Clarity (e.g., 1/7 in place of $.\overline{142857}$)

C. Facility (e.g., converting fractions to decimals for easier comparison)

D. Consistency (e.g., giving all measurements in centimeters)

Rewriting Use Class A: Constraints

Situational constraints are of at least two types. Hardware constraints are those dictated by machine limitations. For example, newspaper typesetting machinery can't deal with exponents or subscripts and even on the fanciest calculators the keyboards are limited to just a few dozen symbols. Because of typewriter limitations, the various computer programming languages (BASIC, FORTRAN, etc.) each have special sets of acceptable symbols. For example, most use 34*78 for the product of 34 and 78 and $2^{3.5}$ is entered as 2**3.5 or 2↑3.5 or 2^3.5.

Algorithm constraints stem from requirements in processing or operating on numbers. For instance, to add fractions using the usual algorithm, mixed numbers must be written as improper fractions and equal denominators are needed. Conversion to decimals for computation purposes is common. And, more recently, calculator and computer algorithms may have their own idiosyncratic requirements.

Examples:

1. Many typewriters and computer console keyboards cannot handle subscripts, superscripts, or fractions written above and below a line. Rewrite each of the following expressions in a way that does not require a shift up or down.
   (a) 1.5×10^8 km, the approximate distance from the Earth to the sun
   (b) $a^2 + b^2 = c^2$, the relationship between the legs a and b and the hypotenuse c of a right triangle
   (c) footnote_1, a subscript used to designate a footnote

(d) $X_1 - X_2$, an expression which could stand for the difference of two first components of ordered pairs

(e) $1\frac{1}{2}$, the size change factor in "time-and-a-half" for overtime

(f) $\frac{a}{b} + \frac{c}{d}$, the expression representing the sum of any two fractions

Answers: Each part has many possible answers.

(a) 150,000,000 km or 150 million kilometers.

(b) a ** 2 + b ** 2 = c ** 2 is done in some computer languages; aa + bb = cc would be correct in algebra, though it is seldom used.

(c) footnote (1) is not uncommon.

(d) X(1) - X(2) is used, as is X1 - X2.

(e) 1.5 or 3/2.

(f) a/b + c/d, or possibly (a/b) + (c/d).

Comment: The popularity of word forms (as in part (a)) is in part because these words are easily typed. The answer to (c) could be "three-halves". The keyboard constraints make calculators and computers more amenable to decimals than to fractions.

Comment: Until recently, most direct printing (as opposed to reproduction of copies) was done by typing or typesetting. This is rapidly becoming less true with the use of thermal print on heat sensitive paper, laser printers, ink sprayed as dot formations, dot-matrix printers, and so on. These newer printers are often under computer control and often don't have the same constraints as have governed the earlier printers.

2. Many calculators have an x^y key. On these calculators, what numbers should take the place of x and y in order to calculate: (a) $\sqrt{10}$, the length of a side of a square whose area is 10; (b) $\sqrt[3]{50}$, the length of a side of a cube whose volume is 50.

Answers: (a) x = 10, y = .5; (b) x = 50, y = 1/3 or .333...to as many decimal places as appropriate in a given problem.

Comment: The radical sign notation for $\sqrt{10}$ and $\sqrt[3]{50}$, and fractional exponents, as in $10^{1/2}$ and $50^{1/3}$, may become obsolete, due to hardware constraints. Neither would be missed by book publishers, since they cause considerable difficulties in typesetting. Such changes in notation would not be unusual in the history of arithmetic. Of the current symbols =, +, −, ×, ÷, and $\sqrt{\ }$, the oldest are + and −, first used in 1489 by Johann Widman.

3. A cornerstone gives MDCCCXCVI as the year of construction for a building. How old is the building?

   Answer: Rewrite MDCCCXCVI as 1896 and then calculate.

   Comment: The rewriting is essential in order to use the standard subtraction algorithm. It is also useful for ease of understanding in today's world (see Section B).

4. Each of the following might appear on a calculator or in a computer program or output. Rewrite in more familiar notation.

   (a) 1.324E+09, a population estimate.

   (b) 3*4.29, a cost.

   (c) 5∧3, a volume.

   Answers: (a) 1.324×10^9, or 1,324,000,000. The E stands for "exponent of 10".

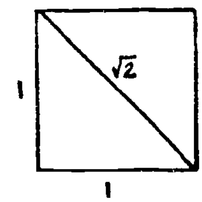
   (b) 12.87. The * stands for multiplication.

   (c) 125. The ∧ stands for powering.

   Comment: The notations in (a) and (c) are forced by the necessity to type on a single line; (b) is forced because X stands for a variable and so cannot also stand for multiplication.

5. A farm 1 mile square is purchased for the purpose of building an airport. To calculate the length of the largest possible runway (to the nearest 200 ft), what rewriting is needed?

   Answer: In theory the length of the longest runway (ignoring width) is $\sqrt{2}$ miles. The $\sqrt{2}$ needs to be rewritten as a decimal and the resulting quantity converted to feet.



   Comment: Many estimates require a particular notation. For example, the idea of significant figures is associated with decimals, so to round to a certain number of significant figures requires changing to decimals.

6. A babysitter is to get \$1.50 per hour and works 3 hours and 30 minutes. To calculate what to pay the sitter, what renaming is needed?

   Answer: One must change the time to hours. Some would change to 3.5 hours and multiply the two decimals. We would change to $3\frac{1}{2}$ hours, multiply 1.50 by 3 and then by $\frac{1}{2}$ and add the products.

   Comment: Changes such as this one are so easy to do, we often forget that renaming is involved.

Rewriting Use Class B: Clarity

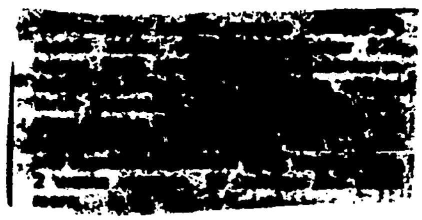
By clarity we mean ease of understanding. For given situations, some ways of writing numbers convey information more clearly than others. For measurements, decimals tend to be clearer. However, if the numbers are very large or very small, scientific notation is often preferred. For ratios, fractions or percentages tend to convey ideas most clearly. For certain growth situations, exponential form (such as 2^{10}) may be most clear.

Similarly, some ways of denoting quantities are more easily understood than others. For most situations, a half mile is easier to comprehend than 2640 feet, and 5 mg of vitamin C in a tablet is easier to understand than .005 g. Given no other constraints, units in quantities tend to be chosen so that the numbers for a given situation will be small whole numbers or simple fractions.

The clarity of any idea to someone obviously depends greatly on the background of that someone. The metric system is clearer than the English system of measurement to most people in the world but not at present to most people in the United States. In many occupations, use is made of special notation and language that may be impossible for someone outside of that occupation to understand without instruction.

Examples:

1. This want ad (from the Chicago Tribune, June 1, 1980) includes 6 numbers. Give a reasonable alternate way of writing each of the numbers or quantities. If you think your alternative is clearer, indicate why.



Answer: 6 month = 26 week; 37 1/2 hours = 37.5 hours; 100% = full; 50% = $\frac{1}{2}$ = half, 2 weeks = 10 days; 10 holidays = 2 work weeks of holidays. In all cases, the ad's notation is clearer except possibly for $\frac{1}{2}$. However, 50% looks larger than $\frac{1}{2}$ or half.

Comment: Ad writers normally are very careful in selecting notation so that the information will be presented in a good light, but the ad must be clear so as to avoid charges of misrepresentation.

2. Indicate an alternate way of reporting each bit of information. (You may wish to perform operations.)

   (a) Three of seven legislators voted for the bill.

   (b) 43,212 of 96,049 registered voters went to the polls.

   (c) Twelve of 24 miles of paving have been completed.

Answer: (a) "The vote was 4 to 3 against the bill" might be clearer if in fact only 7 legislators were involved, and the actual numbers of votes (say 42 out of 98) might be reported if the "three of seven" is intended as a ratio.

(b) With such large numbers, reporting a percentage might better convey the information: "Forty-five percent of the 96,049 registered voters went to the polls."

(c) Since the quotient is a simple fraction one might say "Half of the 24-mile paving is completed."

Comment: Notice the variety of alternatives for the same type of situation.

3. Why is the word "quadrillion" used in this quotation?

> At this very moment, for instance, your brain is so incredibly busy that it processes an estimated 10 to 100 quadrillion impulses a second. (From R. Kotulak, Chicago Tribune, 6/1/80)

Answer: Written as a decimal, ten quadrillion is 10,000,000,000,000,000; 100 quadrillion is 100,000,000,000,000,000. It is shorter and clearer to use the English word.

Comment: If exponential notation were more common or if this article had appeared as a science article, the author might possibly have written "10^{16} to 10^{17} impulses per second." Exponential notation is, however, difficult to set up on newspaper type. Also, because "quadrillion" is to most people a synonym for the indefinitely large quantity "zillion", the author may have chosen the word for impact alone.

Comment: The words billion, trillion, quadrillion, etc., refer to different numbers in Great Britain, than in the U.S. In Britain, 1 billion = 1 million million = 10^{12}, 1 trillion = 1 million billion = 10^{18}, 1 quadrillion = 1 million trillion = 10^{24}. So a London newspaper would have to rewrite this quote to convey the same information to its readers.

4. Why do we say "half-dollar" but write "50¢" (fifty cents) or \$.50, rather than $\$\frac{1}{2}$?

Answer: We use the form that communicates most clearly. Half dollar is easy to say and understand; people are more accustomed to \$.50 or 50¢ in print than to $\$\frac{1}{2}$.

Comment: In some places "four bits" is spoken slang for 50¢, just as "two bits" for 25¢ and "six bits" for 75¢. These are well understood verbal remnants of a monetary system that no longer exists. The "bit" stood for a Spanish or Mexican piece of silver worth 1/8 of a dollar at that time. Notation that communicates clearly with many people sometimes has obscure origins.

5. Give alternate forms of writing the numerical idea "tripled" in this sentence and comment on relative clarity of those alternatives. "The number of calculators in homes has tripled over the past few years."

Possible Answers: "is 3 times what it was a few years ago"

"is 300% of what it was a few years ago"

"is 200% more than it was a few years ago"

Comment: Because of the confusion of the second and third of these, the given or "tripled" seems the best alternative.

6. One nanosecond is equal to 10^{-9} second. Give other ways of writing this.

Answer: Instead of scientific notation, one could write "one billionth of a second" or ".000000001 second".

Comment: The former of these is often used. The latter is not used because it is difficult to keep track of so many zeros.

7. The table below gives various components of U.S. energy consumption in 1977 in three different notations: millions of barrels per day of oil equivalent, percentage of total consumption, and the units commonly used by specialists in the various industries that supply energy. How does each column contribute to clarity of communication?

U.S. ENERGY CONSUMPTION, 1977

SOURCES	Millions of Barrels per Day of Oil Equivalent	Percentage of Total	Quantity in Units Commonly Used for Each Source
Petroleum	18.4[a]	50%	18.4 million barrels per day
Natural Gas	9.2[b]	25	19.2 trillion cubic feet per year
Coal	6.7[c]	18	625 million tons per year
Nuclear	1.3	4	251 billion kilowatt hours per year
Hydro	1.1	3	230 billion kilowatt hours per year
Total	36.7	100%	

[a] Includes imports of 8.7 million barrels per day (mbd), or 47 percent of total oil consumption; excludes 0.2 mpd of exports.
[b] Includes imports of 0.5 mbd oil equivalent, or 5 percent of total natural gas.
[c] Excludes exports of 0.6 mbd oil equivalent.
Source: Department of Energy, Energy Information Administration, *Annual Report to Congress, Volume III, 1977* (Washington, D.C.: Government Printing Office, 1978), pp. 5, 23, 51, 145.

(From R. Stobaugh et al., Energy Future, Baltimore Books, 1979, p. 13.)

Answer: Specialists with each of these energy sources habitually use the particular units given in the column on the right, and may find it difficult to communicate with colleagues in any other terms. The middle column is clearest for showing relative dependence on various sources, apart from quantities used. The left column of numbers expresses all the uses in the same unit so that readers can use the data without learning several specialized vocabularies. The unit chosen millions of barrels per day of oil, is the one most likely to be familiar to someone not in the field.

Comment: The editors of the book from which this table comes give this explanation of why they made the choices they did. (From Energy Future, p. 12)

> **We have yet to meet an energy specialist who can keep all the barrels of oil, trillion cubic feet of natural gas, tons of coal, gigawatts of electricity, and quads of energy in his (or her) head. We certainly can't. In order to keep the energy sources straight, and to make clear the comparisons, we have used barrels of oil per day as our basic unit—though, of course, following convention and common sense as the energy source dictates. Thus, the aim of Table 1-1 is to lay out the equivalences in barrels per day, in addition to showing U.S. energy consumption in 1977.**

8. Prices in many stores in Tijuana, Mexico, just across the border from San Diego, California, are marked in U.S. dollars rather than Mexico pesos. Why might business people there choose dollar notation?

   Answer: One obvious reason is to communicate clearly with American tourists, but dollar pricing is used even in stores frequented mainly by Mexicans. We were told that the proximity to the U.S. border, many local citizens working each day in the U.S., and relative remoteness from the rest of Mexico had resulted in a local economy with dollar values at least as clear to many local people as peso values.

   Comment: In the "duty free shops" found on ships and many airports, items can often be purchased in a variety of currencies, requiring salespeople often to convert from one monetary unit to another. But for clarity, prices are typically shown in one currency.

9. Many scientific calculators enable one to choose among three notations for expressing angle measure: degrees, grads, and radians. 90 degrees = 100 grads = $\frac{\pi}{2}$ radians. Name an occupation in which each of these units is preferred.

   Answer: Degree measure is used in navigation, meteorology, and surveying because its familiarity to people makes it easiest to understand (clarity). Grads are used in some engineering applications; computations with the resulting decimals are much easier than with degrees-minutes-seconds. Radians are both conceptually clearer and computationally simpler when dealing with the sine, cosine, tangent, and other functions as functions of real numbers, as used by mathematicians.

Comment: As the answer suggests, choice of notation in a field is based not only on clarity, but facility also.

Rewriting Use Class C: Facility

Given a particular task to perform with numbers, it is natural to change notations to the best notation for that task. Each of the common notations for representing numbers remains with us because there are certain tasks for which that notation works well. Decimals are easy to order, add, or subtract. Fractions are easy to multiply and divide. Percentages allow ratios to be easily compared. Scientific notation enables easy multiplication or division of very large numbers.

Similarly, one often rewrites quantities to enable easier use later. The strongest argument in favor of converting quantities to metric units is that the metric system is easier to use in most applications.

These reasons for rewriting quantities or numbers are grouped under the heading facility.

Examples:

1. The following quote is taken from an employee's retirement booklet.

> "In planning your budget for the year, remember that your regular pension contribution increases from 1-2/3% to 5% of your salary after you earn $6,600 each year."
> (University of Chicago Supplemental Retirement Plan, 1981)

Calculate the pension contribution of a part-time employee who makes $6000 in a given year. What rewriting did you do?

Answer: Typically, one would rewrite 1-2/3% as 5/3% and then possibly as $\frac{5}{300}$ or $\frac{1}{60}$. This makes calculating the pension easy. The contribution is $\frac{1}{60} \times \$6000$, or $100.

Comment: The choice of decimal or fraction can depend upon whether the decimal is finite or terminating. In the case of 1-2/3%, the decimal equivalent is .01666..., which is difficult to work with, so one often looks for a fraction.

Comment: There are calculators in which one can enter percents in such a way as to avoid rewriting.

Comment: The language of the quotation is ambiguous with respect to the contribution for an income over \$6,600. Does 5% apply to the entire income or only that portion over \$6600?

2. In running, a standard U.S. length is 440 yards (= 1/4 mile). The metric distance closest to this is 400 meters. Which is longer?

Answer: To compare 440 yards to 400 meters, some rewriting is necessary. One way to do the computation is as follows:

$$\begin{aligned} 1 \text{ inch} &= 2.54 \text{ cm} \\ \text{so } 36 \text{ in.} &= 91.44 \text{ cm} \\ \text{i.e., } 1 \text{ yd} &= 91.44 \text{ cm} \\ \text{so } 440 \text{ yd} &= 40233.60 \text{ cm} = 402.336 \text{ m} \end{aligned}$$

and so 440 yd is longer, by a little more than 2 m.

Comment: Because comparison is very easy once quantities are in the same units, this question involves almost entirely the notion of rewriting.

Comment: Many U.S. tracks are built for 440 yd races and have had to be converted to the metric distances because races run in the yard distances (except for the mile) are not recognized for international records. The answer to this question indicates that the metric distance is about a runner's stride less than one lap on such tracks.

3. In 1964, Lyndon Johnson was elected president by having won 486 of the 538 electoral votes. In 1980, Ronald Reagan was elected president after winning 489 of the 538 electoral votes. Both elections were considered overwhelming victories. Reagan received 43,000,000 of 84,000,000 votes cast; Johnson got 43,000,000 of 71,000,000 votes cast. Which was the bigger landslide?

Answer: The answer depends upon whether electoral votes or popular votes are considered as determining landslides. Reagan won $\frac{489}{538}$ of the electoral vote, clearly larger than $\frac{486}{538}$. However, he only received 51% of the popular vote, compared to 61% for Johnson. Reagan backers would rather point to the 91% of the electoral vote than the 51% of the popular vote.

Comment: The particular numbers of these problems enable comparison to be made without substantial rewriting. However, the difference in percentages between the popular and electoral votes is why many people favor abolishing the U.S. electoral vote system, while others defend it as giving a more clear mandate.

4. For which of the following tasks with fractions does converting to a common denominator help? (a) addition, (b) subtraction, (c) multiplication, (d) division, (e) ordering.

Answer: All but (c). Since (a) and (b) are common, we illustrate with (d) and (e). To divide 2/3 by 4/5, change to fifteenths:

$$2/3 \div 4/5 = 10/15 \div 12/15 = 10/12$$

Since 12/15 is larger than 10/15, 4/5 is larger than 2/3.

Comment: Switching to a common denominator is akin to measuring with a common unit, and enables one to concentrate on numerators. Some have suggested teaching division of fractions this way, so as to increase understanding.

5. In 1976-77, there were about 77,000 elementary schools in the United States. An experimental program is tried out in 5 schools and found to be successful. How many orders of magnitude difference is there in attempting to implement this program in all of the schools of the U.S.?

Answer: To calculate order of magnitude, convert to scientific notation, divide, and look at the exponent of the nearest power of 10.

$$\frac{77{,}000}{5} = \frac{7.7 \times 10^4}{5 \times 1} \approx 1.5 \times 10^4$$

There are about 4 orders of magnitude difference.

Comment: Notice that if the program had been tried in only one school, there would be 5 orders of magnitude difference. This agrees with the rough meaning of order of magnitude, namely that each order of magnitude indicates a qualitative difference in the complexity of the situation. For example, implementing in 10 schools is quite different than implementing in 1 school, perhaps like the difference in increasing implementation from 50 schools to 500.

6. A person runs 40 yards in 8 seconds. A second person runs 40 meters in 8 seconds. Car speeds are usually in miles per hour or kilometers per hour. Is it easier to convert the runner's speed to units of car speeds in the English system or the metric system?

Answer: In each case we have to convert 8 seconds to hours, so there is no difference in this regard. However, conversion of yards to miles requires using the number 1760, while conversion of meters to kilometers requires using the easier number 1000. Thus this conversion is easier (for people who know how to convert in both systems) in the metric system.

Comment: When the metric system was first proposed in the late 18th century, there were those who wished to place time on it as well, creating minutes of 100 seconds and hours of 100 minutes, for example. The proposal would have required a changing of the length of the second and to our knowledge has never been implemented. (A day of 25 such hours could be envisioned, we suppose.)

7. In city A, there is precipitation on 75% of the days. In city B, there is rain or snow on an average of 5 days each week. In which city is there more likely to be precipitation on a given day?

Answer: City A.

Comment: It is hard to compare the likelihoods without rewriting. We rewrote the ratio in city B as $\frac{5}{7}$, and then as .714... It is easy to compare that to 75%.

Rewriting Use Class D: Consistency

Consistency may be considered as a precursor to both clarity and facility. It is generally felt that numerical information that is presented in the same format is clearer and easier to work with than information that is given in different forms. We tend to prefer all numbers as decimals or all as fractions to having some of each. Consequently, when information is being gathered or presented, it is customary to decide upon a notation (all chapter numbers in Roman numerals, for example) or unit (measure the lengths in centimeters) in advance or to change to the same notation or unit before doing anything else. Thus many of the examples of the two preceding sections may be considered as illustrating consistency as well.

Aesthetic considerations often lead to consistency even when clarity and facility are not prominent. One often wishes to have numerical information in a consistent format just for the sake of neatness, elegance, or simplicity.

Custom may be informal but may force conformity to a particular way of presenting numbers and units. For instance, one seldom sees a sentence begin with a numeral. Style manuals develop out of a combination of custom and desires for clarity and facility but are established primarily so that presentations will be consistent.

Examples:

1. In Chicago, a block is usually $\frac{1}{8}$ mile. A teacher asked her students to write on their homework papers how far they lived from school. Here are some of the answers she got: 2 miles, 3 blocks, $\frac{1}{2}$ mile, "I live

across the street from school," .8 mile by car. How might the teacher present this information to the class with consistent notation?

Answer: The teacher might convert the information to decimal parts of a mile, to the nearest $\frac{1}{8}$ mile, or to blocks.

Comment: "Three blocks" might be easier for the children to deal with than ".375 mile" or "3/8 mile," so blocks might be the best unit but .8 mile would be difficult to convert.

2. How many quantities and how many intervals are in this excerpt from "Ice Pond", by John McPhee, The New Yorker, July 13, 1981, pp. 92-94? What style regulations are being followed?

> A prototype pond was tried in the summer of 1980. It was dug beside a decrepit university storage building, leaky with respect to air and water, that had cinder-block walls and a flat roof. Size of an average house, there were twenty-four hundred square feet of space inside. Summer temperatures in the nineties are commonplace in New Jersey, and in musty rooms under that flat roof temperatures before the ice pond were sometimes close to a hundred and thirty. The 1980 pond was square—seventy-five feet across and fifteen feet deep. It contained a thousand tons of ice for a while, but more than half of that melted before insulation was applied: six inches of dry straw between sheets of poly-ethylene, weighed down with bald tires. Even so, the old building was filled most of the time from June to September with crisp October air. Something under seven tons of ice would melt away on a hot day. None-theless, at the end of summer a hundred tons remained. "It's a nice alternative to fossil fuels," Robert Socolow commented. "It has worked too well to be forgotten."

Answer: Eight numbers and four intervals. The numbers: 2400(ft^2), 130, 75 (ft), 15 (ft), 1000 (tons), 6 (in.), 7 (tons), 100 (tons). The intervals: year 1980 (twice), nineties, more than half. The New Yorker style manual calls for word forms in articles for all numbers except dates.

Comment: As unusual (and difficult) as this style is, it is still followed by many publishers of fiction.

3. A baseball player gets 10 hits in 40 at-bats. What is the customary way of writing and saying the batting average of this player?

Answer: The batting average is a rate and is calculated by dividing number of hits by number of at-bats. This gives $\frac{1}{4}$ or .25.

However, custom is to write a 3-place decimal for the batting average, .250 in this case, and to say "two hundred fifty" (as if there were no decimal point).

Comment: Having three decimal places for all batting averages looks nicer in tables.

Comment: The same customs apply to winning percentages. A team that wins 20 of 25 games has a winning percentage of .800, spoken as "eight hundred". See Chapter 14, Section D, Example 2.

4. Rewrite the amount $1152.00 as it would customarily be written on the second line of a personal check.

Answer: Eleven hundred fifty-two dollars and no cents.

Comment: There are many variants to the above answer that are just as good. Here are two: One thousand one hundred fifty-two dollars and no/100 cents. Eleven hundred fifty-two dollars and 00/100 cents. Generally, the placement of the word "and" before the "fifty-two" is considered poorer usage but would be accepted by banks.

Summary

The same piece of numerical information can be expressed in a number of ways. Rewriting is an important skill, since no notation suffices for all uses. For numbers, the most common notations are decimals (e.g., 3, .375), fractions (e.g., 3/1, 3/8), words (e.g., three, three-eighths), percents (e.g., 300%, 37 1/2%), or scientific notation (e.g., 3.75×10^{-1}). Within these notations there is further richness because every fraction is part of an infinite equivalence class of fractions, decimal forms can be expanded at will by use of leading or ending zeros, and many word forms are possible. In addition to the notations available for numbers, any quantity can be expressed in quite a variety of ways, by changing units.

We find it convenient to classify into four use classes the reasons people rewrite numbers or quantities.

A. Constraints: Rewrite to fit hardware or algorithm requirements.

B. Clarity: Rewrite to make the information easier to understand.

C. Facility: Rewrite to make the information easier to use.

D. Consistency: Rewrite to make data look nicer to conform to standard practice in a given discipline or to stylistic requirements.

Of course, a particular change of notation may satisfy several of these purposes all at once.

Pedagogical Remarks

All understanding of arithmetic is closely linked to the notations in use for recording and operating with numerical information. For this reason alone, power and flexibility in the uses of arithmetic require the ability to exploit the choices one has among notations for numbers and quantities. Fortunately, rewriting numbers and quantities occupies a significant place in school arithmetic. Important skills include the changing of fractions to decimals, converting units, simplifying fractions, finding equivalents for percents, and writing numbers in scientific notation. Even the "figuring out" of numerical expressions like 39 × 4.3 could be considered as a kind of rewriting.

Getting started. First, students need to realize that the same number or quantity can be denoted in a variety of ways. One can begin with exercises to build up flexibility.

(A) Express 1 mile in as many different ways as you can.

Answers include: 5280 feet, 1760 yards, half of 2 miles, 1609.344 meters, 1.609344 km.

(B) Express 6/8 in as many ways as you can.

Answers include: 3/4, 9/12, 66/88, .75, 75%, 1/2 + 1/4, six-eighths, etc.

Second, teachers must counteract the notion that there is just one right answer to most problems and only one correct way of writing that answer. The child who gets 17/9 as an answer may actually have a more useful way of writing his answer than the child who gets $1\frac{8}{9}$, despite the fact that many teachers prefer the latter way. For a given situation and for managerial convenience, the teacher may wish to insist on a particular notation, but such restrictions should not be imposed for longer than a few days. We think variety ought to be encouraged rather than penalized.

Basic facts. There are basic facts in rewriting just as there are basic facts for multiplication. Here are four we consider as most important:

(1) The ability to write decimal and percent equivalents for fractions with denominators 2, 3, 4, 5, 6, 8, 9, and 10, and probably also for 7, and 11. For example, students should consider 3/8, .375, $37\frac{1}{2}\%$ and 37.5% as completely interchangeable.

(2) The ability to give as many fractions equal to a given fraction as requested, e.g., given 16/10, to come up with 8/5, 24/15, etc.

(3) The ability to switch to and from powers of 10 (10^1, 10^2, 10^3, etc., and 10^{-1}, 10^{-2}, etc.) to the corresponding word forms (ten, hundred, thousand, and tenth, hundredth, etc.), and to the base 10 decimal notation (10, 100, 1000, etc., and .1, .01, etc.).

(4) The application of the previous facts to translating in and out of scientific notation.

Constraints. Calculators and computers force consideration of hardware constraints. If a calculator cannot multiply fractions, then given a situation requiring multiplication of fractions, you must convert to decimals to perform the multiplication. Algorithm constraints are also obvious. For examples, the standard addition and subtraction algorithms ("borrow", "carry", etc.) work only with our standard place value notation, and addition of measures requires that they be converted to the same units.

Clarity. Bring in a newspaper. A number dealing with the national budget might be written as 90.3 billion dollars rather than \$90,300,000,000. Ask students why. Can students think of better ways? With this and similar examples, show that books and articles tend to use the clearest notation.

Facility. To bring this idea home, begin with the following exercises, designed to verify flexibility of notation when doing operations.

> You know $\frac{1}{2} = .5$ and $\frac{1}{4} = .25$. If $\frac{1}{2}$ and $\frac{1}{4}$ are multiplied, will the answer equal the product of .5 and .25?
>
> If 100 cm is converted to inches and 1 meter is converted first to feet, and then to inches, will the answers be equal?

We have found many students who are surprised that the answers to both questions are "Yes", indicating that the substitution of equal quantities in doing operations is not obvious.

Class discussion of the following questions can point out that what is easy for one person may not be considered easy by a second.

> Which is easier, multiplying by .25 or multiplying by 1/4?
>
> Which is easiest, dividing by 5, multiplying by $\frac{1}{5}$, or multiplying by .2?

One furlong, 220 yd, and $\frac{1}{8}$ mile are all the same distance; which would be easiest to compare with a quarter-mile?

Consistency. The desire for consistency is easy for students to understand, for they have had many teachers who require answers to be in some consistent form. Graphing information often brings a desire for consistency. For instance, gather some information from the class that is likely to be expressed in different units, e.g., How long has it been since you last saw a doctor? The answers may be in days, weeks, months, or years. One will usually convert to a common unit when graphing this information.

Consistency and ease of use often go together. To order some numbers, it is useful if they are written in a consistent manner. For example, to order the five fractions from smallest to largest, 2/3, 3/5, 5/8, 9/14, and 7/10, one could convert them all to decimals or rewrite them all with the same denominator.

The most crucial concept to teach is that no notation suffices for all uses. Whether for ease of understanding, ease of operation, consistency, or forced by constraints, it is important that the user of arithmetic be acquainted with a wide variety of notations. A fine way to alert students to the varieties of notation is to discuss the history of arithmetic. (A classic book for this purpose is Florian Cajori's A History of Mathematical Notations, La Salle, IL: Open Court Publishing Co., 1974.) This is a nice way to overlap arithmetic with social studies. The development of ever more suitable and efficient mathematical symbol systems is a hallmark of civilized peoples.

Questions

1. What rewriting is necessary to answer the following question: If the interest on the national debt of the United States is $100 billion a year, how much interest is the government paying out each minute?
2. With a 5-day work week, a person spends 5/7 of her days at work. (a) Convert the fraction to a percent. (b) Comment on which is clearer, the fraction or the percent, and why.
3. Rubik's Cube can be arranged in over 43,200,000,000,000,000,000 ways. There are 86,400 seconds in a day. (a) Convert these numbers into scientific notation and use properties of powers to answer the question: If a person makes one move a second, at least how long would it take the person to cover all of the possible arrangements of the cube. (b) In your opinion, what is the most appropriate unit for the answer, and why?
4. The weights of newborn babies are measured in grams but parents are usually told the weight in pounds and ounces. (a) Why is the conversion done? (b) A baby at birth weighs 3200 grams. Name three other ways in which this weight could be written. Tell which you prefer, the original or one of the other ways.
5. Gasoline use efficiency of cars is measured in miles per gallon in the U.S. and in liters per kilometer in some other places. Notice that not only is there the English-metric difference, but one is given in capacity per distance, the other in distance per capacity. Why is the order switched?

6. Resolve this argument between two people A and B.

   A: You can't add apples and oranges.

   B: But 3 apples + 4 oranges = 7 pieces of fruit.

7. In United States money dealings, the following are synonomous: a quarter, 25¢, and $0.25. Give a situation for each of these notations in which that notation would be preferred over the others, and indicate why the notation would be preferred.

8. (a) Perform the following computation by switching to a common denominator.

$$\frac{3}{8} \div \frac{3}{4}$$

   (b) Perform the following calculation by converting the fractions to percents. Does this make the computation easier?

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{4} + \frac{1}{10}$$

   (c) Perform the following calculation by converting the fractions to decimals. Does this make the computation easier?

$$\frac{11}{10} + \frac{4}{100} + \frac{32}{1000}$$

9. Find an article from a newspaper or magazine in which a change of notation could be used to help clarify the arithmetic of a situation.

Notes and Commentary

1. The chapters of Part III
2. The term "maneuver"
3. Use classes for maneuvers
4. Alternate use classes for maneuvers
5. Related work for Part III and Chapter 11

1. The chapters of Part III. Chapters 11-13 are ordered by the degree of adjustment one makes with the data. In rewriting, one does not change the number or quantity, but changes how it is represented. (Implicit understanding of the number-numeral distinction is required.) In estimating, a number or quantity is used either because it is the best available in a situation or is close to a second number or quantity. (Understanding of ordering and closeness are required prerequisites.) In transforming, numerical information is replaced by other information which may not look at all like the original. Chapter 14 discusses alternate representations of data and in that sense is most akin to Chapter 11; however, the techniques used in arriving at these representations involve ideas from each of the preceding chapters.

2. The term "maneuver". We agonized long over a term that would fit the four processes that we wished to discuss. A first draft used the term adjustment as a global term for Part III, and the chapters were given the following corresponding titles: 11. Changing notation; 12. Micro-adjustments; 13. Macro-adjustments; 14. Changing mode of presentation. Perhaps the reader can see why this term was abandoned, for it seemed only to obscure the content. Furthermore, both transforming and displaying seemed to do more with numerical information than merely adjust it.

We considered the following terms as possible alternates: treatments, strategies, tactics, manipulations, modifications. "Treatments" was felt to be too medical and too often used in other contexts. "Strategies" and "tactics" are part of current problem solving jargon and do not usually refer to the kinds of mathematical tactics discussed in Chapters 11-13. "Manipulations" are more associated with the operations of Part II. "Modifications" was considered too weak, too connotative of small change.

We also considered verb forms: massaging, fondling, processsing, handling. Statisticians are particuarly fond of using the terms "massaging" or "fondling" data, but there is a sense of exploration that surrounds these terms that is not always present in real situations. Processing and handling were considered too vague.

Maneuver as a global term came to us quite late but was immediately appealing. One of its definitions exactly suits the idea behind this part: "an adroit move, skilfull proceeding, etc., esp. as characterized by craftiness" (The Random House Dictionary, Unabridged Edition, p. 871). Skillful handling of numerical information is a craft, and while it does not require stealth as some maneuvers do, it requires wisdom, experience, and perspicacity.

To call rewriting, estimating, transforming, and displaying "maneuvers" carries a message we consider most appropriate: A person does not change notation or estimate or scale or display merely for the lack of anything else to do. There is a reason. The term maneuver suggests the existence of such a reason.

Another attraction of the word was its dual appearance as both noun and verb. Thus it conveys not only the process (rewriting is a maneuver) but also the action (maneuvering by estimating).

One criticism of this word, that it has its origin in military actions, seemed to us to be weak. Strategies and tactics carry that same origin but have come to be used in discussions of problem solving without that connotation. We hope the same occurs with maneuvers.

3. Use classes for maneuvers. We discuss here the global organization for Part III. Discussion of individual chapters and the use classes follows the chapters as usual for Chapters 12-14 and follows this note for Chapter 11.

Unlike either of Parts I or II, each chapter in this part has the same use classes. The summary at the end of Part III displays, as a consequence, the following matrix.

		Reasons			
		constraints	clarity	facility	consistency
Maneuvers	rewriting				
	estimating				
	scaling				
	displaying				

That is, when one maneuvers numerical information without calculating new data (such calculations would fall under

Part II), one is either forced to do it, does it for ease of understanding, does it for ease of use, or does it to be consistent internally or with external criteria.

As with the use classes in Parts I and II, in a given situation, the reasons for maneuvers often overlap. A common overlap involves constraints. Many situations require some maneuver of a given type. For instance, data is given in a variety of notations and one wishes to type this onto a table. So we must rewrite and the situation fits into the rewriting use class called constraints. But which maneuver to use is generally dictated by considerations of clarity, facility, or consistency.

4. Alternate use classes. It was suggested to us that economy or efficiency might constitute a fifth reason for doing these maneuvers. We felt that clarity and facility were reasons that often substituted for efficiency or speed or economy of processing or use.

A sixth reason for doing a maneuver might be aesthetics. It may look nicer or be more elegant to rewrite decimals as fractions, or to use an estimate, or to employ a scale value, and particularly to graph or picture numerical information. We felt categorization into aesthetics would be difficult and omitted this category.

We realize that our organization may be oversimplified, and we offer some questions others may wish to consider. The questions are followed by some brief comments on them.

(A) Could the other parts have been organized so that all chapters in them had the same use classes? The answer is Yes for Part I, which could have been organized so that each of its chapters had the same use classes as Chapter 1. In doing this, we would have concentrated on the numbers that were components of the n-tuple, elements of the set, and values of the variable rather than on the n-tuple, set, or variable itself. For example, the interval in "temperatures are expected in the 80s tomorrow" would be classified as location rather than as depicting a neighborhood use of set. We chose not to operate at the level of individual numbers for two reasons: first, this ignores the mathematical object--set, n-tuple, or variable; second, we would have oversimplified the notion of uses to such an extent that this higher level would be missed.

(B) Is there some explanation, other than coincidence or convenience, for there being the same use classes in Chapters 11-14? Again, yes, and the explanation is most evident by considering a context much broader than mathematics. Why do we do anything? (E.g., why do we usually wear shoes?) The answers tend to fall into three categories: ease of use (It's easier to walk with shoes over some types of ground.); custom (It's considered a part of civilized dress.); necessity (You can't dine in certain establishments without footwear.). These categories parallel the categories we entitle facility, consistency, and constraints.

If we consider contexts in which communication is involved, then clarity is an important attribute. Thus the four use classes of this chapter emerge rather naturally from consideration of broad rationale for human behavior. This suggests that the broad organization of Part III is only a rough way of analyzing the reasons why these mathematical processes are undertaken.

(C) What alternative organizations of use classes could have been chosen? The careful reader might look closely at the organization within sections and the examples and specify the use classes more closely to the particular mathematics utilized. For example, in Section D of Chapter 11, situational constraints are either hardware or algorithmic. Are these the only ones? An alternative organization could be more specific.

Another alternative is to organize by the particular mathematical processes rather than by collections of processes. For instance, why do we rewrite into scientific notation (rather than why do we rewrite)? This alternative would be easier to adapt into current schoolbooks. However, it tends to make one lose sight of the more general principles at work.

5. Related work for Part III and Chapter 11. We know of no other work that has attempted to organize what we call maneuvers in a way comparable to that given here.

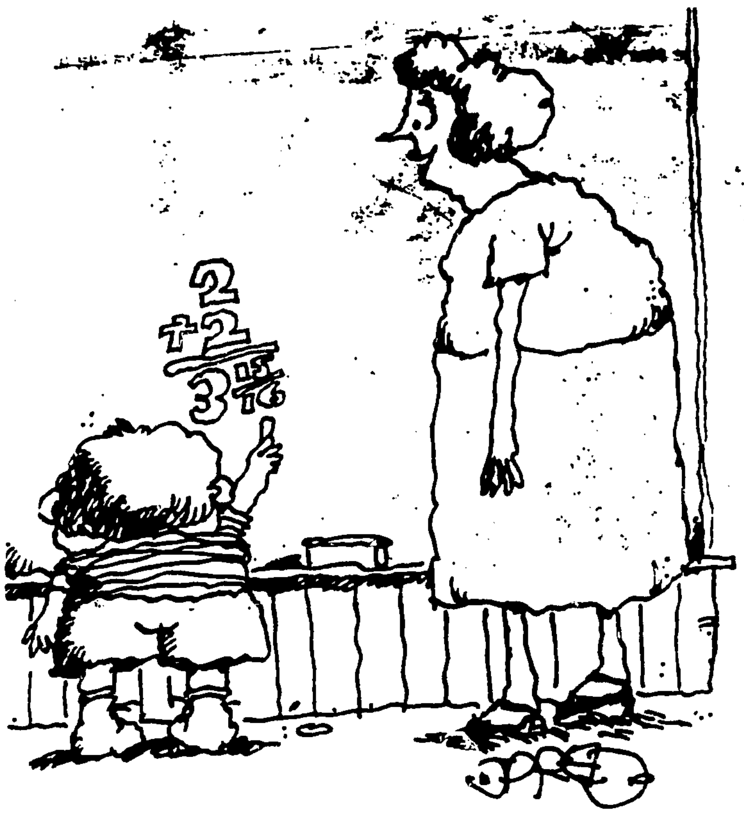
Chapter 11 owes homage to the number-numeral distinction in the form given by Max Beberman et al. in the University of Illinois Committee on School Mathematics materials of the late 1950s. The concept of renaming certainly did not originate with Beberman, but the authors were made aware of its broad implications through this work.

Bell's "everyman" list includes "existence of many equivalence classes" and "flexible selection and use of appropriate elements from equivalence classes (e.g. for fractions, equations, etc.)." Selection from an array of equivalent choices is much the same as what we call here rewriting, as indicated by his discussion of that idea:

> Through a long series of experiences starting in kindergarten (or before), a youngster should come to know that most mathematical things come in many equivalent forms and that much of school mathematics deals with conversion from one form to another. He should also realize that problem solving both of textbook exercises and of real life problems frequently involves recognition of equivalence plus good judgement about which of a number of possibilities are appropriate for use in a particular situation. (For example, 3 + 7 and 10 are each sometimes useful; 5 minutes to 3 p.m., 2:55 p.m., and 14:55 all express the same time; in calculating 1/2 +1/3, 3/6 is a more helpful form than 1/2; 2x - 11 = 3x + 7 may be a fairly direct translation as a mathematical model of a problem

situation, but $-18 = x$ is a more convenient version in the end. (Bell, 1974, p.198)

We know of no others who have attempted to catalog or categorize the uses to which rewriting is put. We guess that others have considered rewriting quantities simultaneously with rewriting numbers but do not have references to cite on this either. We would appreciate being informed of related work.



"That's very close, Scott."

CHAPTER 12

REASONS FOR ESTIMATING AND APPROXIMATING

Arithmetic (as well as the rest of mathematics) is commonly depicted as involving only the most exact of ideas. But in fact the numerical information we use often involves estimates. An estimate may be a single number or a neighborhood (see Chapter 3, Section C). Sometimes an estimate is to a fixed known value, as in the approximation $\frac{22}{7}$ for π, but sometimes an estimate is used when there is no fixed or known value, as in an estimate of the temperature in a city or the anticipated cost of a trip.

Estimating is often viewed as a task one engages in as a less attractive alternative to dealing with exact numerical information. The uses of estimating verify just the opposite, namely that estimating is often more reasonable than avoiding estimates and that estimating is often the only choice one has in a situation.

An entire chapter could easily be devoted to ways to estimate and types of estimates. These include rounding (up or down to the nearest whatever), successive closer and closer approximations, estimates found from polling or sampling statistical measures (mean, median, mode, etc.), and the various indices (e.g. the Dow-Jones stock averages) used to estimate economic conditions. Although examples of all these are found in this chapter, the purpose of the chapter is to identify the reasons for using estimates.

The use classes for estimating are identical to those found in the other chapters of Part III:

Constraints, e.g., the attendance at a future political rally must be estimated.

Clarity, e.g., 4 billion dollars may be easier to comprehend than 3.986 billion dollars.

Facility, e.g., in buying a number of $19.95 items, for many purposes computing with $20 will be easier.

Consistency, e.g., body temperatures are always rounded to the nearest tenth of a degree.

Estimating Use Class A: Constraints

In a large number of situations, there is no choice but to estimate, because exact values are not obtainable. Here is a brief typology of such situations.

1. A number may be unknown, forcing an estimate at best. Predictions of the future, guesses about the past, estimates of military strength of countries or economic conditions of competitors, and even educated guesses regarding what groceries will cost are estimates forced by lack of knowledge.
2. A quantity may be different each time it is measured, forcing one to estimate it. Temperatures, populations, air pressures, and typing speeds are common examples. Situations involving relative frequencies, such as a number of heads in 100 tosses of a coin, are of this type.
3. Physical measurements are, for the most part, not exact. For example, no sheet of paper is exactly 20 cm long. (Look under a microscope and the edge of this paper is seen to be quite rough.) Thus many measures that seem exact are more accurately viewed as estimates close enough for practical considerations.
4. The cost of obtaining an exact value may be prohibitive in time or money, so estimates are substituted. Pollsters sample to estimate how many people watched a particular TV program or how many prefer a candidate; they don't ask everyone.
5. A quantity makes sense only as a whole number, so any other result from a computation must be adjusted. For instance,

if candy is 3 for 10¢, because there are no coins for parts of a penny, a person will pay 4¢ for a single piece.

6. Safety or a necessary margin for error may dictate a substantial overestimate. When going to the store, the shopper estimates what items will cost and then overestimates to play it safe. In designing a new building, safety factors are expected to meet more than minimal requirements.
7. A value may itself have been calculated from estimates. If one can only estimate the number of people available for a job, then the time they will take to do the job must also be an estimate.
8. The number is not in a form that lends itself to computation. For example, to add irrationals such as π or $\sqrt{2}$ to rationals, one must use an approximation.

In the examples of the next sections, estimates are better than exact values, but there still usually is a choice between the two. In the above types of situations, there is not a choice whether to estimate; estimates are necessary. People who believe that estimates are inferior to exactly calculated answers are ignoring the substantial number and variety of situations in which it is wiser or necessary to use estimates.

Examples:

1. Each of the statements below is taken from The World Almanac, 1979 edition, and contains an estimate. For which of the eight reasons mentioned above is each estimate necessary?
   (a) "Current theories say the first hominid [human-like primate] was Ramapithecus, who emerged 12 million years ago."

(b) The area of the Sahara desert is 3,320,000 sq mi. [The area of the United States is just over 3,600,000 sq mi.]

(c) The average July temperature in San Diego, California is 69°F.

Answers: (a) The actual value is unknown (type 1 above).

(b) The area varies (type 2) and there is a theoretical limit to the ability to measure it (type 3). Furthermore, the cost of obtaining anything close to an exact value may not be worth the additional accuracy (type 4).

(c) The temperature varies as years go on (type 2).

Comment: Sometimes several considerations combine to make an estimate necessary, as in (b).

Comment: An analysis of data that appears in almanacs shows estimates to be much more common than many people seem to realize.

2. If a school puts an upper limit of 25 students in each English class, how many classes will be needed for 110 students?

Answer: 5, rounded up from 4.4

Comment: The more general term "adjustment" is more appropriate than "estimate" for this kind of situation. 4.4 is adjusted up to 5.

Comment: This example falls under Type 5, for one can only have a whole number of classes.

3. If fruit is being sold for 3 at 40¢, what is the likely price for one piece of fruit?

Answer: Not $\frac{40}{3}$ ¢, but 14¢.

Comment: Grocery stores almost always round fractions of a cent up. In this case, if a kindly merchant were inclined to round down to 13¢, it would make it senseless to sell 3 for 40¢.

4. A coin believed to be unbiased (balanced) is to be tossed 100 times. (a) Give a single number estimate into which the most likely number of tails that will occur. (b) Give an interval estimate into

which the number of tails that will occur will tend to fall at least 90% of the time.

Answer: (a) 50, calculated by multiplying the probability of tails (1/2) by the number of repetitions of tossing (100).

(b) Probability tables indicate that, for 100 tosses of a fair coin, the number of tails will be from 41 to 59 about 90% of the time.

Comment: In (a), 50 tails is the maximum likelihood estimate, but exactly 50 tails is still not very likely. Tables of the normal distribution are used to determine a precise answer to (b).

5. Each of the questions below requires an estimate for an answer. For which of the eight reasons in this section is the estimate necessary?

(a) How long does it take to get to the nearest airport from your home?

(b) What is the population of the People's Republic of China?

(c) What is the width of a human hair?

(d) How much pressure should a car safety belt be able to withstand?

(e) What will be the price of a Quarter Pounder at McDonald's one year from today?

Answers: (a) 2; (b) 2 or 4 [the 1979 estimate was 953,578,000]; (c) 3 or 1 [the seldom used unit called the hairbreadth is defined as 1/48 of an inch]; (d) 6; (e) 1.

Comment: The numerical answers to questions (a), (b), and (c) are more reasonable as intervals than as single numbers. This is true of most examples for reasons 1-4 given in this section.

6. A standard dimension for a roll of wallpaper is 16 yd long and 2.21 ft wide. How many rolls of wallpaper are needed for a room 8' from floor to ceiling and walls with a total width of 42'?

Answer: In theory, 8' goes 6 times into 16 yd. In practice, this would only occur if there were baseboards to give a margin for trimming. If the latter is the case, then each roll can be said to cover 6 x 2.21 ft or 13.28 ft of floor width. Thus 3 rolls will not cover 42 ft and 4 rolls are needed (with quite a bit left over).

Comment: In decorating a house or apartment, judicious estimation of floor, wall, and shelf needs almost always involves estimating of this type. The same holds in sewing. Overestimation is much preferred to underestimation because more of the fabric or covering may not be available, or may come from a different dye lot. If there are patterns that need to be matched, even greater overestimation might be desired.

7. At least five of the six numbers in the following quote are necessarily estimates. Name these five numbers and indicate why estimation is necessary. [Source: The Book of Lists #2, by Irving Wallace et al., New York: Wm. Morrow and Co., 1980, p. 142.]

"A 1200-lb horse eats about 15 lb of hay and 9 lb of grain each day. This amounts to 1/50 of its own weight each day, or 7 times its own weight each year. The real gluttons in the animal kingdom are birds, who consume more than 90 times their weight in food each year."

Answer: 15 lb, 9 lb, and 90 are necessarily estimates because the numbers in question vary. The 1/50 and the 7 are estimates forced by these earlier estimates. Though 1200 lb might be deemed a necessary estimate because of the inability to measure exactly, it seems here more used for facility, to ease the other calculations, or for clarity, to make the paragraph easier to understand.

Comment: Animals' food needs are closer to being proportional to surface area than to weight, because food intake (measured in calories) is balanced by heat loss through the animal's surface (also measured in calories). The apparent gluttony of birds is accounted for by their large surface area for their weight.

8. In calculating the amount of cement needed for a circular play area 5 yards in diameter and 4 in. deep, what estimates are necessary and why?

Answer: Though the play area may be called circular, it has thickness so is a cylinder. The volume of a cylinder is given by the formula $V = \pi r^2 h$, where in this case $r = \frac{5}{2}$ yd and $h = 4$ in. For consistency of units, rewrite 4 in. as $\frac{1}{9}$ yd. Then

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi\left(\frac{5}{2}\right)^2 \times \frac{1}{9} \text{ cubic yards} \\ &= \frac{25}{36}\pi \text{ cubic yards} \end{aligned}$$

Estimation is now necessary because the above value is difficult to compare with the units in which cement is sold. (Use $\frac{22}{7}$ for π, so $V \approx \frac{550}{252}$ cubic yards. If one divides with a calculator the result is 2.1825397. This too is an estimate, forced because the answer is an infinite repeating decimal.

Comment: A person should order $2\frac{1}{4}$ or $2\frac{1}{2}$ cubic yards of cement.

9. In Example 8, we used a TI-35 for calculating a decimal approximation to $\frac{550}{252}$. It gave 2.1825397. A Sharp EL-211 gives 2.1825396. A Texas Instruments SR-51A gives 2.182539683. Why do we get different values?

Answer: The decimal for $\frac{550}{252}$ is $2.18\overline{253968}$. This forces an approximation to be built into the calculator chip. The TI-35 rounds to 8 decimal places, the SR-51A rounds to 10 places, while the EL-211 truncates after 8 places.

Estimating Use Class B: Clarity

A school budget of \$148,309,563 for a school population of 62,772 pupils might be reported as "about \$150,000,000 for 63,000 pupils." A house on a lot whose width was surveyed as 40.13 feet would almost certainly be said to be on a "40-foot" lot. In these cases, the estimate is <u>clearer</u>, that is, easier to understand, than the more precise actual figure. Estimates for clarity are almost always calculated by rounding.

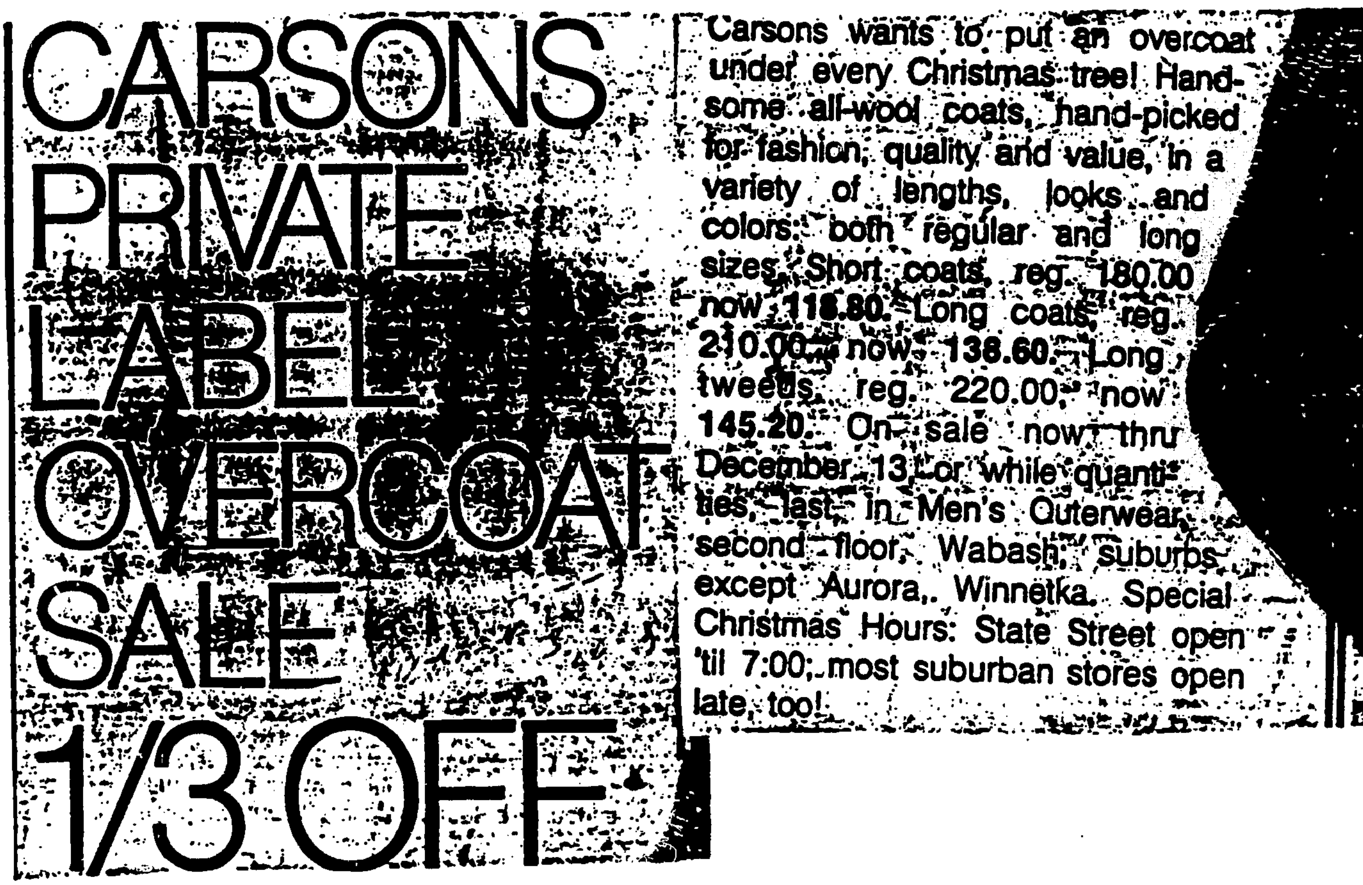
<u>Examples</u>:

1. Indicate how each of the following might be estimated for purposes of clarity. (Each number is taken from actual reports.)
   (a) 8'11.1", the height of Robert Wadlow, reportedly the tallest person who ever lived
   (b) 3,851,809 sq mi, the area of Canada
   (c) 2574 mi, the air distance between New York and San Francisco
   (d) 4,493,491, the reported number of scouts and leaders in the Boy Scouts of America (1980)

   Answers: (a) 8'11"; (b) 3,850,000 sq mi; (c) 2600 mi, (d) 4.5 million

   Comments: All of these numbers are necessarily estimates, so here we estimate the estimates. One almost always estimates large numbers, as in (b) and (d), perhaps because the short-term memory of most people can retain only a few digits. (c) would be estimated for convenience and simplicity. (a) is least likely to be estimated because of a desire to keep records to as much accuracy as possible.

2. Examine the advertisement reproduced below. (a) What number is an estimate? (b) What is the actual value being estimated?



Answers: (a) 1/3; (b) The actual discount is 34% for each of the three types of coats mentioned.

Comment: 34% is not as easy for most people to understand as 1/3, which seems to be the reason for using the estimate. Carsons could not be accused of misadvertising because they have understated the amount of the discount.

Comment: Why Carsons would use a 34% discount instead of 1/3 is another question. A possible reason is that .34 is more conveniently entered into calculators.

3. To facilitate conversion to the metric system, some have proposed that the most used metric measures be given temporary names that suggest their English measure counterparts. A meter would be called a "metric yard" under this idea. If people acted as if this metric yard were an actual yard, there would be an error of about 3.37" in every 36", or about a 9% error. (a) What might be called a "metric quart"? Estimate the error (to the nearest percent) if people acted as if the equivalence were exact. (b) Do the same for a "metric pound".

Answers: (a) The liter is a natural for the metric quart. Since 1 liter = 1.057 liquid quarts, there would be about a 6% error. (b) A metric pound might be 500g. Since 1 pound = 453.59237 grams exactly, there is about a 10% error if 500 grams is called equivalent.

Comment: In track, 1500 meters has long been called the "metric mile" despite 1600 meters being much closer (1 mile = 1609.344 m exactly) because races are run over 1500 m and not over 1600 m.

Comment: Our inclusion of this example does not constitute endorsement of the idea. It may be that such verbiage would create confusion rather than clarify the new units.

4. What single number estimate is often used as a measure of the wealth of a community?

Answer: Median family income in the community and per capita property value are two common estimates.

Comment: Per capita property value is a rate calculated by dividing total property value by the population. Median family income is a better estimate than the mean (average) family income because the former is less affected by the wealth of a few people.

Comment: Though a single number estimate is more easily understood than giving lots of data in such a complex situation, much information can be lost. In this example, either of the suggested answers misses intangibles (e.g., community pride or physical beauty of the surroundings), services provided from outside the community (e.g., by state or federal governments), and some capital assets (e.g., values of governmental buildings and parks).

5. National TV newscasts always include the Dow-Jones Industrial Average (DJIA) change for the day and the closing average. What does the DJIA estimate?

   Answer: The DJIA is based upon the worth of the stocks of 30 "industrial" companies, including General Motors, U.S. Steel, and IBM. It is viewed as a measure of the economic health of U.S. industry.

   Comment: There are other stock averages, some including many more stocks than the DJIA, but due to its long history and the amount of analysis done with it, the DJIA is more familiar, and thus a clearer estimate to many people.

6. On four tests of basic arithmetic facts, Jill scored 8 out of 10, 9 of 12, 9 of 9, and 7 of 15. Estimate what percent of basic facts Jill knows.

   Answer: One might possibly estimate by adding the numbers of questions and the numbers of correct responses. She answered 33 of 46 questions correctly, or about 72%. However, if the tests are on four different areas and in each area there are the same number of basic facts, then one might calculate individual percentages and take their mean for the estimate. The percentages are 80%, 75%, 100%, and about 47%, with a mean of about 75%.

   Comment: The differences between these estimates is not enough to make a practical difference. Scores on tests are always estimates of knowledge, and having a single score for Jill is merely easier to understand.

Estimating Use Class C: Facility

When considering how many $3.98 items you can afford to buy, it is easier to do your calculations as if they cost $4. In planning a business trip and comparing relative costs of driving versus flying somewhere, it is more realistic to use estimates and the calculations are a lot easier: "The trip will be about 800 miles, my automobile gets about 20 miles per gallon, and gasoline costs about $1.40 per gallon; but there will be two days extra driving and the motel will cost $35 and meals about $30. On the other hand, the cheapest way to fly costs about $300, and I'll need to rent a car at the destination for five days at $40 per day since I won't have mine..."

It is obvious that it is often easier to calculate using estimates than using exact figures. It is less obvious that even with calculators and computers taking the work out of computation, estimating may make things a lot easier with no important loss in the quality of the answers. If fact, answers derived using shrewd estimates may be more reasonable and more realistic than those that attempt to be exact.

Examples:

1. The Internal Revenue Service allows taxpayers to round all amounts up or down to the nearest dollar, a practice called "dollar rounding". Give two reasons for this practice.

   Answers: The roundings up and down tend to balance out, so there is little difference in totals to be paid. There is little lost by this practice, and the ease of working with two less significant figures saves the IRS time and money. It is also easier for the taxpayer and means fewer mistakes on tax forms.

Comment: In accounting practice generally, and particularly in banks, accounts are made to balance to the penny. When one of the authors worked for a bank, he was told that being meticulous in that way sometimes turned up serious errors (e.g., mistakes in crediting accounts) that went well beyond the pennies involved.

2. Suppose one decides that federal anti-poullution funds should be allocated in relation to the areas of regions of the United States. California has an area of 158,693 sq mi. The states of the Northeast have areas as follows: Maine--30,920; Vermont--9,609; New Hampshire--9,304; Massachusetts--8,257; Rhode Island--1,214; Connecticut--5,009; New York--49,576. Which should receive more money under this policy, California or all the states of the Northeast?

   Answer: California. The easy way to determine this is to round to the nearest thousand. This gives 31, 10, 9, 8, 1, 5, and 50 thousands, the sum of which is 114,000, far less than the area of California. Only if the sum was very close to 158,000 would it then be reasonable to add the more precise areas.

   Comment: If anti-pollution funds should be allocated on the basis of population, the Northeast would receive more funds than California.

3. Calculate the amount of cement needed for a sidewalk 5' wide, $3\frac{1}{2}$" deep, and 30' long. (The 2-by-4 used for sidewalk forms is actually $1\frac{1}{2}$" by $3\frac{1}{2}$", accounting for the $3\frac{1}{2}$" depth.)

   Answer: V = length x width x depth

   $= 30' \times 5' \times 3\frac{1}{2}''$

   To ease calculation:

   $\approx 30' \times 5' \times \frac{1}{3}'$

   $\approx 50 \text{ ft}^3$

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Cement is sold by cubic yards ($27 \text{ ft}^3 = 1 \text{ yd}^3$).

$\approx 2 \text{ yd}^3$

Comment: Both estimates ($\frac{1}{3}$ ' for $3\frac{1}{2}$ " and 2 yd^3 for $\frac{50}{27} \text{ yd}^3$) make the situation easier to deal with. The choice is made to overestimate each time to provide a margin of safety (see Section A).

4. In preparing a $25,000 budget for a year for a family of four, to what nearest amounts should estimated expenses in various categories be rounded?

Answer: $100 would be reasonable. It is easy to work with and is only .4% of the total. ($100/$25,000 = .004).

Comment: Carrying more accuracy than this is seldom worth the effort, and leads to frustration when records are not obtainable or when expenses exceed the budget by small amounts that are of little importance in the total picture.

Comment: There is a view held by many adults that dealing with estimates in such situations is a weakness and that one should strive for exact amounts at all times. In our opinion, such a view is misguided.

Estimating Use Class D: Consistency

Government reports on the percent of people unemployed always give that percent to the nearest tenth. For instance, if 8 million of 99 million potential workers were unemployed, the government would report 8.1% unemployed rather than 8.08% or any closer approximation to 8.$\overline{08}$%. Here an estimate is forced because the original data are inexact, but the particular choice to report in tenths is done so as to be consistent from month to month and consistent with the precision of the original data. Similarly, results of calcuations from physical measurements should be recorded with significant digits consistent with the input measurements. Consistency in estimates also arises from conventions or regulations, or from a desire to have data or uniform appearance in tables, charts, or graphs.

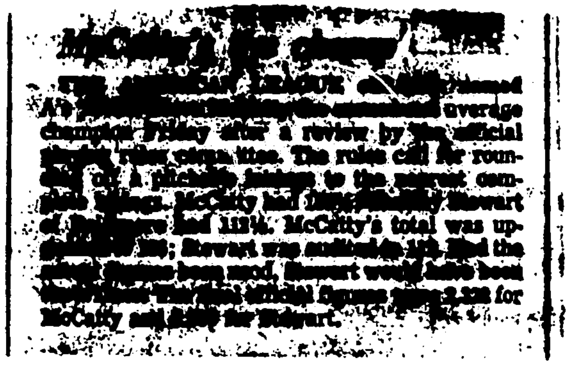
Examples:

1. A team's winning "percentage" is traditionally reported as a decimal rounded to the nearest thousandth. Give the winning percentage of a team that has won 5 of 16 games.

   Answer: An exact value is $\frac{5}{16}$ or .3125. This would be rounded to .312 or .313 for reporting purposes.

   Comment: For the particular numbers, an exact decimal could be given. In contrast, winning 4 of 14 games would give a percentage of .$\overline{285714}$. Because the decimals are so different, it makes sense to impose a traditional rounding criterion.

2. The following news item appeared in the Chicago Tribune on October 10, 1981. It refers to the 1981 Major League baseball season.



This is an example of rounding to be consistent with rules. State another rule that would be equally as consistent.

Answer: We can think of three other consistent procedures.

(1) Truncate or round down to the nearest whole number of innings. (2) Round up. (3) Use the exact data.

Comment: The formula for earned run average (ERA) is:

$$\frac{\text{ERA}}{9} = \frac{\text{no. of earned runs allowed}}{\text{no. of innings pitched}}$$

From this equation, one can calculate that McCatty allowed 48 earned runs and Stewart 29. Thus before the rounding, if the other procedures are followed, McCatty and Stewart have the following ERA's (rounded to three decimal places), respectively: (1) 2.335 and 2.330; (2) 2.323 and 2.310; (3) 2.327 and 2.323. In all of these cases, Stewart would have been the winner.

3. Indicate the normal precision to which each of the following quantities is reported.

(a) the inflation rate for a given month

(b) the mpg of a new car

(c) total U.S. government defense expenditures

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Answers: (a) to the nearest tenth of a percent; (b) to the nearest mile per gallon; (c) to the nearest billion or tenth of a billion (hundred million) dollars.

Comment: Experts in many fields may utilize greater precision than novices. Those working with (a) or (c) would need to use greater precision; (b) is calculated from data that is unlikely to give an answer that is a whole number.

4. A rectangular lot is 39.5 meters long and 24.3 meters wide. To achieve consistent accuracy, how should the area of this lot be reported?

Answer: Multiplication of length by width yields 959.85 square meters. Since the given information has only three significant figures, an area of 960 m^2 should be reported.

Comment: The given dimensions 39.5 and 24.3 are themselves estimates. Any value between 39.45 and 39.55 m would have been reported as 39.5 m, and any value between 24.25 and 24.35 m would have been reported as 24.3 m. Thus the calculated area could range between 956.6625 m^2 and 963.0425 m^2. This further justifies the reporting of 960 m^2 as the area, and only two significant digits should be trusted.

Comment: If the 0 were significant in 960, scientists would use scientific notation 9.60×10^2 m^2.

5. A gasoline pump and an automobile odometer indicate that 8.5 gallons of gas have been used in travelling 240.2 miles. What is the mpg?

Answer: A calculator gives 28.258824 mpg, which should be rounded to 28 because 8.5 has only two significant figures.

Comment: $\frac{240.25}{8.45} = 28.43...$ and $\frac{240.15}{8.55} = 28.08...$, indicating that to use a third significant digit in the answer is not called for in the data.

Comment: Neither the odometer nor the "filling" of the tank is likely to have the indicated accuracy to tenths, giving further reasons to round.

6. The Natural History Magazine of March 1974 gave the following estimates of top speeds for animals (in mph):

Cheetah	70
Elk	45
Rabbit	35
Giraffe	32
Elephant	25
Squirrel	12
Pig	11
Giant tortoise	.17
Garden snail	.03

What kind of consistency was used in arriving at these estimates?

Answer: Consistency in there being two significant figures (with the possible exceptions of the snail and the cheetah).

Comment: Leading zeros for decimals or trailing zeros for whole numbers often make obscure how many significant figures are intended. For example, if a measure is given as .0035 m it could be interpreted as having two, three, or four significant figures, as could a measure given as 1500 meters. One way to attend to that ambiguity is consistent use of scientific notation. For example, 3.5×10^{-3} m indicates two significant figures but 3.500×10^{-3} m indicates four significant figures; similarly for 1.5×10^{3} m and 1.500×10^{3} m.

Summary

The uses of estimating may be categorized into the same four use classes as the uses of rewriting.

A. Situational constraints: Estimate because of the unknown nature of the data, the prohibitive cost of obtaining an exact value, the incompatability of the quantity, the measuring unit, variability of a quantity, and theoretical measurement restrictions.

B. Clarity: Estimate (usually by rounding) to make a given piece of information easier to understand.

C. Facility: Estimate to ease comparison, computation, or other future use.

D. Consistency: Estimate to conform to desired or customary precision, significant digits, or to have consistent data for tables or charts.

Pedagogical Remarks

Estimation in schoolbooks. In textbooks, students are often asked to estimate the answer to a problem before they do the precise calculation. For instance, a student is asked to estimate an answer to a problem, say 49 x 82. (The idea is to round 49 to 50 and 82 to 80, then multiply to get the estimate 4000.) However, when this type of problem is first given to students, they usually don't estimate. They will multiply out 49 x 82, getting 4018, and either use 4018 as the estimate or round that answer to get an estimate. The teacher tells the student that he or she has done it backwards. "You are supposed to estimate first, then calculate. Your estimation can help you to check whether the calculation is correct." Most students are unconvinced. They have a sure-fire way to guarantee a good estimate.

It has often been said that there is logic even behind wrong answers and "wrong" procedures. That is the case in the above scenario, for indeed the student's procedure is more like the real world than the teacher's advice. Seldom do we estimate when we can easily get an exact answer. Furthermore, we often calculate for the purpose of obtaining an estimate. These are often the only estimation activities students encounter. With only such activities, students don't learn when estimation is an appropriate maneuver.

Basics of estimation. For estimation, students must be able to round up, round down, round to the nearest multiple of 10, 100, 1000, .1, etc., and even round to the nearest multiple of some arbitrary number. (For example, if you need a certain number of eggs, it is useful to be able to round this number to the next higher dozen or half-dozen.) Students should be able to do simple arithmetic with powers of 10, e.g., $10^2 \times 10^3 = 10^5$ and $10^2 \div 10^3 = \frac{1}{10}$ or 10^{-1}, and to rewrite as noted in the pedagogical comments to Chapter 11.

Students should be able to round fractions to whole numbers. In a recent National Assessment study, students were asked to estimate the answer to the fraction sum

$$\frac{8}{9} + \frac{12}{13}.$$

Of course the idea is to notice that each individual fraction is very close to one, so the sum is quite close to 2. This was a very difficult problem for most students. We believe it was difficult only because of the refusal of teachers to work on simple estimation.

Older students should know estimates to various irrationals such as $\sqrt{2}$, $\sqrt{3}$, and π. For π they should know all of the common estimates, including 3.14, 3.1416, 3.14159, and 22/7. They should know estimates that relate English and metric measures, such as a kilometer is about .6 of a mile. And they should be able to approximate any fraction by a finite decimal, such as approximating 1/7 by the decimal .14.

Students need to know the consequences of a poor estimate. A good example is the famous prediction by pollsters of the defeat of Truman by Dewey in the presidential election of 1948. Estimates carry with them a margin of error, but often that margin is overlooked. A child's score on a standardized test is an estimate of the child's ability in the area of the test, but seldom is the score treated as an estimate.

It is very important that students learn that estimation is often the preferred alternative or the only alternative possible in a situation. It is not by chance that the section of this chapter dealing with situational constraints is the longest of the four sections, for constraints force estimating a great deal more than the public realizes or the textbooks suggest. Whereas people are led to believe that estimating is a weak sister to exact computation, the truth is that estimation is quite often the stronger sister or the only child.

Getting started. As we have suggested many times in this book, a good place to start is with a newspaper. Pick up an article that has a good deal of numerical information. Some of that information is almost certain to be estimates. Ask students why the estimates have been used. Ask how they might have been determined.

Constraints. It is quite important to discuss situations where estimates are necessary. Students realize this in the case of future prediction without having to be reminded. A more interesting set of examples consists of situations where there is a tradeoff between cost and accuracy, as in TV ratings and any other instances of sampling. The point to be made with sampling is that, beyond a certain point, it costs too much to get just a little more accuracy. (The relevant saying is "Penny wise, pound foolish.")

Clarity. Using estimates for clarity is a matter of judgment. Talk to students about the population (according to the Census) of their community or the nearest big city. Is it helpful to give such an exact figure as the Census gives?

Notice that clarity is distinct from precision. Often the more precise value is not as easily comprehended. For example, it is easier to remember that a kilometer is a little over 1600 yards than to recall or memorize that 1 km = 1609.344 yd. Precision is an important idea and should not be underrated, but there are occasions when sacrificing precision for clarity is a good idea.

Facility. Any easy to understand use of estimation, and one often supported by textbooks, is the notion of replacing numbers by other numbers (the estimates) that are easier to work with when a situation does not require a great amount of accuracy. Begin with the standard problem:

How many $7.95 records can be bought for $50?

The first thing to do here is not to divide, but the ascertain what sort of answer is good enough for this situation. Here we note that an obsession with getting an answer to a few decimal places is a great disservice both to the mathematics of the problem (the answer must be a whole number) and to the student, who will waste time. It is without question more reasonable to divide 8 into 50 and get 6.25, which is rounded down to 6, or even to realize immediately without performing the division that 6 will suffice.

The school obsessions with exact answers in arithmetic do a great disservice to people. These obsessions often force unnecessary calculations and keep people from gaining experience with estimation judgments. They can kill intuition with detail (the proverbial losing the forest through the trees) and reinforce the false notion that exactness is always to be preferred to estimation.

Consistency. Estimates made for consistency often are needed in the preparation of printed columns of data. The financial and sports pages of the newspaper, calorie counts in articles dealing with foods, prices of cars, all provide grist for the mill. Duplicate such a column of data and ask students what consistent pattern is being followed. Make up a value more precise than that given. Ask students how it would be rounded to appear in the column.

Questions

1. Give these 1974 World Series batting averages to the customary number of decimal places. Which require no estimating, but merely rewriting?
   (a) Steve Garvey, 8 hits in 21 at-bats.
   (b) Reggie Jackson, 4 hits in 14 at-bats.
   (c) Bill Buckner, 5 hits in 20 at-bats.
   (d) Joe Rudi, 6 hits in 18 at-bats.
   (e) Vida Blue, 0 hits in 4 at-bats.
2. According to the Book of Lists #2 (see Example 7, Section D), "Sales figures for the International House of Pancakes show that their 485 U.S. restaurants sold a total of 63,487,564 pancakes in 1978. On an individual basis, each branch restaurant sold an average of 130,902 pancakes that year." (a) Which of the numbers in the quote are estimates? (b) Use estimates of the costs of pancakes to estimate the toal income the International House of Pancakes chain realized just from the sale of pancakes. Make your estimate to the nearest 10 million dollars.
3. Give a reasonable neighborhood (Chapter 3, Section C) for each of the four single number estimates found in the following excerpt from the Chicago Sun-Times, July 19, 1981: "About 1,500 teaching jobs and 4,000 career service jobs were eliminated by the Board of Education's 1981-82 budget. A Sun-Times comparison of the new budget with the 1980-81 budget also shows that about 125 jobs were added to central administrative and support offices, about a 6 percent increase."

4. According to the 1970 U.S. Census, the population of Seattle, Washington, was 530,831. (a) What do you think would be a reasonable estimate of the actual population (since we know these census figures are indeed estimates)? (b) Why does the U.S. Bureau of the Census publish figures that are more accurate than the situation calls for?

5. A family estimates the cost of a vacation at $1500. This estimate can fall under any of three of the use classes of this chapter. Which three classes are these and why does this situation fit each?

6. Each value below is given as it is reported in The World Almanac, 1979 edition. What would be a clearer estimate?

   (a) The population of Charleston, South Carolina, according to the 1970 census was 66,945.

   (b) Louisians had 2,223,033 licensed car, truck, or bus drivers in 1977.

   (c) It is 4,876 miles from Moscow to Washington.

7. What estimate to π is most appropriate in calculating the area of a circle with radius (a) 6 cm; (b) 6.0 cm; (c) 6.00 cm? (d) Why are different estimates needed for what are mathematically equal radii?

8. Give the type of situational constraint forcing an estimation exemplified in each situation.

   (a) Trying to ascertain the world population on January 1, 2001.

   (b) Guessing the weight of a friend.

   (c) Estimating how many people are watching a presidential press conference.

(d) Calculating how many pieces of candy will be needed for "trick or treaters" on Halloween.

(e) Determining the width of an atom.

Notes and Commentary

1. Semantics related to estimating
2. Types of estimates
3. Precision and Accuracy
4. Significant figures
5. Categories of necessity
6. Work of others
7. Reverse rounding

1. Semantics related to estimating. The word "estimate" is both a verb and a noun, relating both to a process and the result of that process. For instance, the *Random House College Dictionary* defines the verb form of "estimate" as

> "to form an approximate judgment or opinion regarding the value, amount, size, weight, etc., of; calculate approximately"

and the noun form as

> "an approximate judgment or calculation, as of the value, amount, or weight of something." (p. 452)

The word "estimation" is likewise used both to represent the process and the result of estimating. (Similar semantics apply to the word "transform" as it is used in Chapter 13.) The word *approximate* is a synonym for the verb form, and the phrase *approximate number* is used by many as a synonym for the noun form.

These semantics can be confusing. Consider the common estimate 3.14 for pi. As a number, 3.14 is no more or less exact than pi is. When 3.14 is used in a calculation, the calculation itself may be exact. Thus the phrases "calculate approximately" and "approximate number" are in some sense self-contradictory, what rhetoricians call oxymorons.

The semantic confusion stems partially from mixing the reason behind the estimation either with the process of arriving at an estimate or the result of that process. Once one has a single number estimate, it is a number just like any other, and the calculations are as exact as any other calculations.

2. Types of estimates. (A) In some cases, an exact number is known (e.g., the number of students in a school) and the estimate represents a clearer, more consistent in precision, or easier to use, value than the original number. A similar example is the set of decimal approximations to irrational numbers such as pi or the square root of two.

(B) In other cases, no exact number is known (e.g., the number of days next month when it will rain or the weight of a

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typical brontosaurus) and the estimate is necessary. Such estimates may be made by "educated guesses" or calculated from sophisticated procedures. (That many estimates are themselves the result of exact calculations is not often appreciated.)

(C) In between types (A) and (B) are the dealings with measure, where one often has a value good enough for all purposes but still realizes that, in theory, it is an estimate, and may round that value for other purposes. For instance, a weight of 153.02 pounds would seem to be good enough for most purposes but is an estimate and in almost all cases the value 153 pounds would be used instead.

(D) Also in between are those cases where it is known that there is an exact value but that value may be unattainable and so estimates are used. For example, there exist algorithms to calculate all roots of the equation x^4 - x = 1 exactly, but usually one is content with an estimate.

In some places, the word _approximation_ is used when the value is known and _estimate_ is used when it is not. We considered making that distinction but looking at books and dictionaries convinced us that in common usage the semantics are hopelessly muddled.

Furthermore, the processes are often intertwined. A best guess estimate may be the first input to a process of successive approximations to an exact value. And, conversely, an approximation to an exact value may be an input to a best guess estimate of a second value that cannot be determined.

3. _Precision and Accuracy_. Of the four use classes for estimate, the one most difficult to conceptualize is consistency. The notion of consistency of estimates is tied in with notions of precision and accuracy, but associated with these two words are a mix of conceptual and semantic issues.

We begin with the semantics. The _Random House Dictionary of the English Language_ gives different definitions for precision as used by physical scientists and mathematicians. For mathematicians, precision is

> "the degree to which the correctness of a quantity is expressed".

While for physical scientists, precision is

> "the extent to which a given set of measurements of the same sample agree with their mean".

There are also two definitions for accuracy. For mathematicians, accuracy is

> "the degree of correctness of a quantity, expression, etc.".

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For physical scientists, accuracy is

> "the extent to which a given measurement agrees with the standard value for that measurement".

Notice how close the definitions are; the words are often used synonymously by non-experts.

Yet it is clear that measures can be precise without being accurate. For example, a poorly calibrated weighing scale might give the same reading time after time (good precision) but all the readings would be wrong (poor accuracy). In statistics, these qualities are associated with reliability and validity. A test is said to be reliable to the extent that repeated taking of the same test would yield the same score for a given individual. A test is said to be valid if it measures what it purports to measure. Each of these can exist without the other.

Examination of a number of expositions in schoolbooks and books for teachers convinces us that these ideas are quite muddled for many mathematics educators. For instance, in the Thirtieth Yearbook of the National Council of Teachers of Mathematics, we find:

> "Precision is a technical term and is defined by the unit measure used. . . . If you are given two measurements, the one with the smallest unit of measure is the more precise." (p. 247)

We can hardly believe that 3.094 cm is less precise than 10 mm, but it would be under this definition. Nor, if a student were asked to change a measure from 10 yards to 30 feet, can we believe that the change of notation has endowed the measure with more precision.

Utilizing the physical scientists definition of precision and accuracy, there are 4 distinct ideas--repeatability of a measure vs. the truth conveyed by the measure--and these ideas would seem to be teachable. But we gave up on an attempt to follow science practice in using these words because of inconsistent semantics elsewhere. It may be that the words "reliable" and "valid" as used in educational testing would provide a better starting point as <u>labels</u> for the <u>ideas</u> that should be taught.

The ideas conveyed by reliablity and validity (if we were to settle on that vocabulary) lead to obvious questions that do not have easy answers. For example, we usually measure once to find what we hope is a "true" value--we usually have no outside truth with which to compare it. This is often dealt with by stating an interval within which we "expect" the true value to fall and much of statistics deals with quantifying those expectations. Also, if we have just a single measure or set of measures, the question of how well <u>repeated</u> measures agree with one another is moot.

This has long been dealt with in education testing by "split half" methods--pretending the same test has been repeated twice. Recently developed methods of "bootstrapping" in statistics carry this idea much further and make the results more believable.

4. Significant Figures. Vocabulary and semantics aside, better teaching about estimates would be facilitated by better teaching about "significant figures" as an important issue in reporting measures or counts. Better teaching would include more and earlier attention to the issues (the whole topic is often disposed of in a few pages in seventh or eighth grade textbooks) and, especially, discussions in the context of actual count and measure situations, where an intuitive feel for what is "significant" in an everyday sense can be conveyed. Earlier and more careful teaching about significant figures would also make teaching about rounding, scientific notation, and "standards" for measure units easier to teach. This is another of the ways in which mathematics education could profitably be more in tune with science education.

5. Categories of necessity. Of the four use classes of estimating, constraints is without question the most interesting. The six types of constraints given in section A are derived from reasons for sampling given by Wallis and Roberts (1956). Their reasons (pp. 136-139) are: (1) sometimes the measuring process destroys the items; (2) the gain in accuracy from a complete survey may not be worth the cost; (3) making a large number of measurements may lead to poor data; (4) the population may be infinite, as in determining whether a coin is fair (there are infinitely many possible tosses); (5) the population may be inaccessible (something particularly true of historical records). While the details of the reasons are not the same here as in Wallis and Roberts, the intent is similar, namely to point out that what seems to be a weak alternative (for them, sampling instead of dealing with the entire population; for us, estimating instead of using exact values) is in fact a better or unique alternative.

6. Work of others. The literature dealing with estimating is rather extensive, and since the identification of estimation as one of ten basic skills in mathematics by the National Council of Supervisors of Mathematics (1976), the literature has increased substantially. Most of the literature has dealt with methods for estimating, and we are not aware of any others who have examined in depth the reasons for estimating.

Trafton (1978) relates estimation and mental arithmetic. When estimation is used for mental arithmetic, we would classify that as facility, namely ease of computation.

Bright (1976) relates estimation to measurement, even to the point of giving the following quite restrictive definition: "estimating is the process of arriving at a measurement or measure without the aid of measuring tools". This connection with measurements is similar to the connection of estimation and mental arithmetic in the sense that estimation is viewed as a substitute for a (seemingly) more effective or more precise procedure. If one foregoes existing measure tools to make life easier, we might call that facility, while if the absence of measuring tools forces an estimation, we might classify this use under necessity. Some, however, might prefer to create a category entitled efficiency or economy, to incorporate all of those uses in whch estimating is done to provide a quicker way to arrive at an answer to some question.

A review of research on estimation may be found in Reys et al. (1980).

7. Reverse rounding. When there are exactly 700 students in a school, the exact count 700 may be perceived as an estimate because of the zeros. Yet such a happening would be expected in 1 of every 100 counts of large populations. To make the value seem more accurate, one could change it to 701. This process, which we term reverse rounding, has been done in the past, and could be said to constitute a rare use class for estimating that is of questionable value.

We have heard that the first attempt at an accurate value (to the nearest foot) for the height of Mt. Everest came out to be 29,000 feet. This was supposedly changed to 29,002 so that all would realize the accuracy. (Recent measurements give a height of 29,028 feet, so the problem has vanished.)

In the 1970 U.S. Census, according to the 1981 Hammond Almanac, approximately 3700 places had populations of 6200 or more. Of these, one of the authors counted only 27 places with populations evenly divisible by 100 (37 woud be expected) and only 1 (Bergenfield, NJ, population 29,000) evenly divisible by 1000 (4 would be most likely). Is it possible that census workers change figures, say from 16200 to 16201, to explicitly demonstrate significant figures? We do not know. We encourage a more in-depth look at possible examples of reverse rounding.

The 1977 Information Please Almanac does not always give populations agreeing with the 1981 Hammond Almanac, so original sources should be consulted. In the 1977 source, only 1 of the 440 or so cities with populations over 45,000 has a population evenly dvisible by 100 (Minneapolis, 434,000). Again this is fewer than would be expected by chance.

CHAPTER 13

REASONS FOR TRANSFORMING

A student correctly answers 19 questions on a 25-question test. We say that 19 is the raw score. That score may be transformed in a number of ways: to a percentage correct (76%); to a rank in a class (say 5th); to a numerical grade (perhaps 2.7, standing for C+); to a grade level (maybe 8.5); to a standard score (like 63). The results of transforming may bear little or no resemblance to the raw score, but the attribute being described has not changed, namely that each of these numbers (76%, 5th, 2.7, 8.5, 63) is a way of stating the student's score.

The general rule behind a change in a single number or quantity is called a transformation. Most of the examples in this chapter are of one of the following five types of transformations.

a. ranking -giving the position of a number in comparison with other numbers (giving rise to percentiles as well as ranks)

b. shifting--picking a particular quantity to be a zero point and measuring all other quantities from that point (as in measuring heights on Earth from sea level instead of from the center of the Earth)

c. scaling--dividing all quantities by a fixed quantity (as is done to arrive at the Consumer Price Index)

d. log scaling--writing all quantities as a power of the same quantity, then using the exponents (as is done with sound intensity)

e. standardizing--converting raw data to numbers based upon positions in a normal or other standard distribution (as is sometimes done with test scores)

Transforming is done for the same broad reasons as rewriting and estimating, giving rise again to four use classes.

A. Constraints, e.g., in comparing prices over long periods of time some adjustment is necessary to account for inflation

B. Clarity, e.g., ranks may convey a clearer picture than the original numbers

C. Facility, e.g., calculations with scaled quantities may be easier to make

D. Consistency, e.g., putting a number of tests on the same scale allows them to be treated in similar fashion

There are, of course, interactions among these reasons. Because transformations can be more complex than merely rewriting or estimating, one usually transforms only if more than one of these purposes can be met. For example, the scaling of raw scores to obtain their SAT equivalents is done for clearer reporting, allows easier comparison from year to year, and is necessary because not all students take the same test.

Transforming Use Class A: Constraints

In dealing with data gathered at different times or places or under different circumstances, some transforming may be necessary. For instance, in situations where great accuracy in measurement is needed, temperature, pressure, and height above sea level may need to be taken into consideration (hotter bars are longer, objects weigh less as they are higher from the surface of the Earth). In times of inflation, costs in different eras are only reasonably compared if the data is somehow modified by earning power. If one wishes to equate tests that are qualitatively different, some sort of transformation is needed.

A second type of constraint may be imposed by physical or mathematical limitations. A set of data may contain data too diverse to fit on a single graph with linear scales; there exist transformations that can fix that problem. Raw data may not be amenable to particular types of analysis, but yet some transformation of the data may make those analyses possible.

Examples:

1. Adjust the U.S. per capita income values given below (Source: Department of Commerce, Bureau of Economic Analysis) to give per capita income values "in constant 1967 dollars", that is, accounting for the changes in the Consumer Price Indexes (CPI) since that time (Source: Department of Labor, Bureau of Labor Statistics).

Year	Per capita income	Consumer Price Index
1967	$ 3167	100
1975	5851	161.2
1977	7043	181.5
1979	8638	217.4
1981	10517	272.4

Answer: The scaling is done by dividing income by CPI and multiplying by 100.

1967	$3167
1975	3630
1977	3880
1979	3973
1981	3861

Comment: Some scaling, or its equivalent, is necessary to compare buying power in the different years. The result shows that buying power in 1979 was highest of these years and that buying power in 1981 had retreated to below 1977 levels.

2. If wages rise as fast as expenses, then a person's ability to pay stays the same. Thus in a time of changing wages and expenses, a way of comparing prices in different years is to use a scale that indicates how long a person would have to work in order to earn enough money to pay for a particular item. Examine the following table, taken from the Hammond Almanac for 1981, p. 250.

THE CHANGING AMERICAN FOOD BASKET: 1914–1979 Source: U.S. Department of Agriculture

The table indicates the amount of food that could be purchased through the years with one hour of labor.

Item	Unit	1914	1919	1929	1939	1949	1959	1969	1977	1979
Bread, white	Lb.	3.5	4.7	6.4	7.8	9.8	11.1	13.9	15.9	15.8
Round steak	Lb.	0.9	1.2	1.2	1.7	1.6	2.0	2.5	3.2	N.A.
Pork chops	Lb.	1.0	1.1	1.5	2.1	1.9	2.6	2.8	3.1	N.A.
Sliced bacon	Lb.	0.8	8.9	1.3	2.0	2.1	3.3	3.8	3.6	N.A.
Butter	Lb.	0.6	0.7	1.9	1.9	1.9	2.9	3.8	4.2	4.1
Cheese	Lb.	1.0	1.1	1.4	2.5	2.4	3.8	3.4	3.3	N.A.
Milk, fresh	Qt.	—	—	—	5.8	7.0	9.1	11.6	13.5	13.5
Eggs, fresh	Doz.	0.6	0.8	1.1	2.9	2.0	4.1	5.1	6.8	8.0
Oranges	Doz.	—	0.9	1.3	2.2	2.7	3.3	3.8	4.4	N.A.
Potatoes	Lb.	12.3	12.4	17.9	25.1	25.1	34.8	38.9	37.6	N.A.
Tomatoes* (canned)	—	—	2.8	4.4	7.3	9.1	14.1	16.2	15.0	N.A.
Margarine	Lb.	—	1.1	2.1	2.8	4.5	7.8	11.5	9.8	N.A.

* #2 can from 1914 to Sept. 1954; #303 can beginning Oct. 1954.

(a) Compare the numbers of eggs that could be bought with an hour's labor in 1949 with that in 1979. The numbers you must compare are 2.0 and 8.0. What are the units of these numbers?

(b) What does the table generally show regarding trends in prices?

Answers: (a) The units of the numbers are dozen eggs per hour. Thus in 1979 one hour's labor could buy 8 dozen eggs, compared with 2 dozen eggs in 1949. For eggs, the earning power was multiplied by four in these 30 years.

(b) The table generally shows that salaries have been rising faster than food expenses in almost every category of food in almost every decade since 1914.

Comment: Each horizontal row of the table forms a different scale. Each scale is actually no more than a measure in an uncommon food unit (food per hour). Notice how much clearer trends can be seen when using this measure. Comparisons of prices are often done with such scales.

3. Baking at high altitudes requires changes in recipes. The Joy of Cooking, by Irma S. Rombauer and Marion Rombauer Becker (Indianapolis, IN: Bobbs-Merrill Co., 1975) recommends the following changes for an altitude around 5000 feet: "...reduce the double-acting baking powder or baking soda by 1/8 to 1/4 teaspoon for each teaspoon called for in the recipe. Decrease sugar 1 to 2 tablespoons for each cup called for, and increase liquid 2 to 3 tablespoons for each cup indicated. Raise the baking temperature about 25°." Ingredients and baking

temperature for a jelly roll are given here. Modify these for Denver (altitude 5280 feet).

Preheat oven to 375°.
3/4 cup sugar
4 egg yolks
1 teaspoon (tsp) vanilla
3/4 cup cake flour
3/4 tsp double-acting baking powder
1/2 tsp salt
4 egg whites
1/2 cup jelly or tart jam.

Answer: The oven should be preheated to 400°. The sugar should be decreased by 3/4 to 1 1/2 tablespoons. The baking powder should be decreased by 3/32 to 3/16 of a teaspoon (1/8 of a teaspoon is in between). The amount of vanilla (a liquid) is so small it need not be changed.

Comment: Most large cookbooks note alterations in recipes for high-altitude cooking. Baking in Mexico City (altitude 7347 feet) or La Paz (the capitol of Bolivia, 11,916 feet) requires more substantial changes.

Comment: The changes recommended in this question constitute a scaling of the ingredients mentioned and can be accomplished by multiplication. The amount of baking powder can be found by multiplying by 3/4 to 7/8, but the use of two different units (tablespoons and cups) makes finding the factors for sugar and liquid more difficult.

Transforming Use Class B: Clarity

Transformations achieve clarity when they place numerical information in a simpler context. Ranks, percentiles, and stanines may be easier to understand than the raw data that went into making them up. If the numbers used in a measuring situation are quite complicated, a modification of the unit may result in more easily understood numbers.

Transformations also achieve clarity if the scale onto which the numbers are transformed is well-known to the user. For example, to a person familiar with the meaning of a standard score, such scores will be clearer than the raw scores from which they might have been calculated.

In general, the clarity of a situation depends upon the knowledge and experience of the user. Thus one person may prefer a transformation of data while another would feel better dealing with the raw data.

Examples:

1. In 1970, Albuquerque, New Mexico had a population given as 244,501, 58th largest of United States cities. In 1980, its population was 331,767, 44th largest. What do the ranks signify that the raw populations do not?

   Answer: The ranks indicate that many cities in the U.S. are larger than Albuquerque. The change in ranks from 1970 to 1980 shows that Albuquerque is growing more quickly than most large cities in the U.S.

   Comment: In a case where a government program is to give aid only to the largest cities, ranks attain more importance than populations.

2. In school A, Albert scores 30 of 35 on one subtest of a battery of standardized tests. This is in the 74th percentile nationally and the 85th percentile locally. In school B, Betty scored 30 on the same test, which put her at the 62nd percentile locally. What can be deduced from this information?

   Answer: Since Albert was in a better percentile locally than nationally, the students in his school are not scoring as well as students across the nation. Betty and Albert know about the same amount, having scored the same, but students in Betty's school did better on this test than students in Albert's school.

   Comment: Betty may think she knows less than Albert thinks he knows, because they are comparing themselves to populations with different levels of achievement.

3. The population of the United States is about 230,000,000 (1982). What number would be entered into a table if the heading stated (a) population (millions); (b) population (thousands); (c) population (billions).

   Answers: (a) 230; (b) 230,000; (c) .23.

   Comment: This type of transformation is commonly employed to save room in tables without loss of clarity. One divides the raw data, here 230,000,000, by the unit indicated in parentheses. See the comment to Example 4, Section C, Chapter 14.

   Comment: It is also possible to interpret the change from 230,000,000 to 23 with the heading (in millions) as rewriting. See note 4 in the Notes and Commentary section.

4. One can obtain the year number in the Jewish calendar by adding 3760 or 3761 to the year in our normal Gregorian calendar. (Add 3760 before the Jewish New Year in September, add 3761 afterwards.)

Thus January 1, 2000 will be in the year 5760 of the Jewish calendar.

In what way is this latter date clearer to a religious Jew?

Answer: The latter date is measured from the date of Creation as calculated using the chronology of the Old Testament. Thus it has meaning whereas a date calculated from an event in the life of Jesus would not.

Comment: Since years in our time are denoted by numbers in the thousands in almost all calendars, the choice of one over the other is usually not dictated by mathematical considerations of clarity, but more often by religious significance.

5. To a golfer, what is the clearest way to assess how well someone is playing while still on the course?

Answer: Compare the score to "par", the expected score for that number of holes. For example, if a golfer has taken 39 shots on 9 holes whose par scores sum to 36, his score so far is "3 over par". The transformation here is a shift of 3 from the sum of expected scores on each hole.

Comment: The expected score is assigned mainly on the basis of the distance from tee to green, with some adjustment for difficulty of terrain. This is itself a sort of transformation. Golf courses are designed so "par" is more or less comparable from one course to another; for standard eighteen-hole courses par usually ranges from 70 to 73 for men and 72 to 78 for women.

Transforming Use Class C: Facility

It is often easier to work with numbers that have been transformed. For instance, Examples 2 and 3 of Section A exhibit transforming done to equate information obtained from different times, allowing for easier comparison of the data.

Transforming is also done for ease of calculation or extrapolation, particularly when the given data contains very small or very large numbers, or numbers that differ by many orders of magnitude. It is also done to convert data to a simpler scale to use.

Examples:

1. Ease of calculation. To calculate the average of her bowling scores 123, 105, 119, and 109, Jane ignores the hundreds and averages 23, 5, 19, 9, and then adds 100 to this average. Will this work? What kind of transformations have been used?

   Answer: She has shifted the scores 100 down, computed an average, then shifted 100 up. It works, and it is often the easiest way to calculate the average of a set of large numbers that are rather closely spaced.

   Comment: If numbers are all very small or very large, the average can be calculated by the analogous transformations using multiplication and division. For instance, to average .0073, .0610, and .0324, multiply the numbers by 10000, average the resulting whole number products, and then divide by 10000.

2. <u>Facility</u>. The set of ten scores at left below has been transformed in the right column. (a) What transformation has been used? (b) Calculate the mean of the set of numbers at right. (c) Use this mean to find the mean of the set of numbers at left.

Original Scores	Transformed Scores
98	13
98	13
96	11
93	8
87	2
85	0
83	-2
82	-3
78	-7
75	-10

Answers: (a) 85 has been subtracted from each score. (b) 2.5

(c) Add 85 to get 87.5 as the mean at left.

Comment: The number 85 was chosen because it is nearly midway between 98 and 75 , so would make the transformed scores closer to zero. Some would prefer to subtract 80, to make the subtractions easier, or 75, to keep all scores nonnegative.

Comment: Regardless of the number subtracted from all original scores, the set of transformed scores has the same standard deviation as the set of original scores. So this kind of transformation is particularly useful in statistics.

3. <u>Placement</u>. Student scores on standardized tests are sometimes transformed into stanines. A stanine is a number from 1 to 9, calculated from percentiles as follows: 0-4%, stanine 1; 5-11%, 2; 12-23%, 3; 24-40%, 4; 41-60%, 5; 61-77%, 6; 78-89%, 7; 90-96%, 8; 97-100%, 9. (a) What proportion of students will be classified into stanine 5? (b) What proportion will be in stanine 8?

Answers: (a) 20%, (b) 7%. Thus stanines do not appear with equal frequency.

Comment: The transformation of percentile into stanine loses a lot of information, yet many people want to work with 1 digit rather than 2 because it is so much easier for placement purposes and for interpreting the many scores in a battery of a test.

4. Ease of extrapolation. Display the population of California from 1850 (when California entered the Union) through 1970, given the following census data. Estimate the 1980 population from this.

Year	Population	Year	Population
1850	92,597	1920	3,426,861
1860	379,994	1930	5,677,251
1970	560,247	1940	6,907,387
1880	864,694	1950	10,586,223
1890	1,213,398	1960	15,717,204
1900	1,485,053	1970	19,953,134
1910	2,377,549	1980	?

Answer: To display the actual numbers would require an interval from 0 to 20 million and half of the values would be below 2.5 million. So we take the logarithm of the population (using a calculator) and graph that instead. (See next page.)

Year	log (Population)	Year	log (Population)
1850	4.97	1920	6.53
1860	5.58	1930	6.75
1870	5.75	1940	6.84
1880	5.94	1950	7.02
1890	6.08	1960	7.20
1900	6.17	1970	7.30
1910	6.38	1980	?

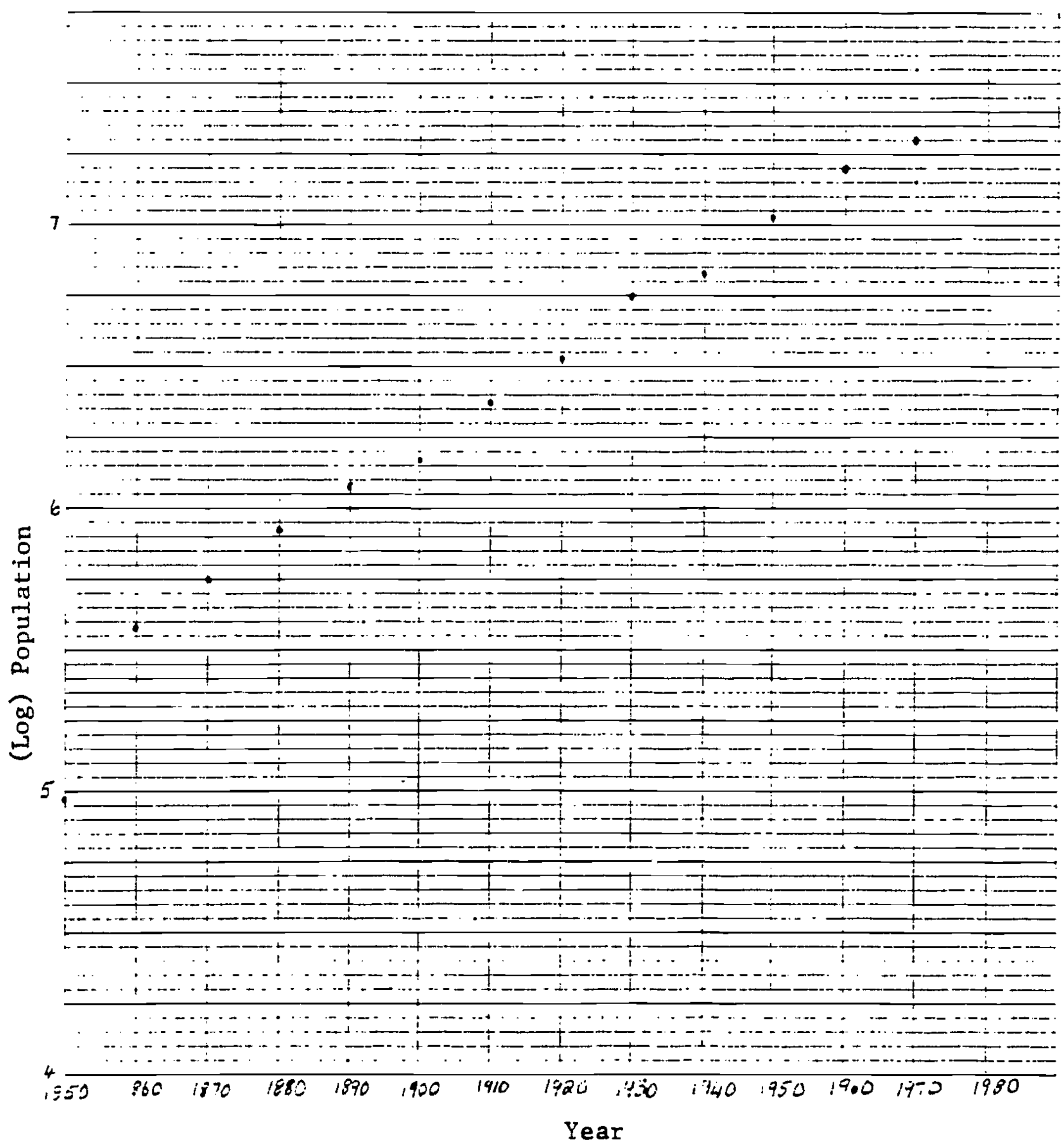
The data suggests log (Population 1980) would be 7.44, corresponding to a population of about 27,500,000.

Comment: Actually the growth rate slowed and the population was about 23,667,565 in 1980, with log(Population) being 7.37.

Comment: The only point that is not close to a line of best fit on this graph is the one for the 1850 population. This reflects the extraordinary rise in population following the discovery of gold in 1849.

Comment: It would be close to impossible to estimate the 1980 population from the graph if the raw populations, rather than their logs, were used, so this example could fit under constraints (section A) as well.

Comment: An alternative is to use semilog graph papger and graph the actual population. Then no transformation is needed on the data, but a compensating transformation has been carried out to create the graph paper.



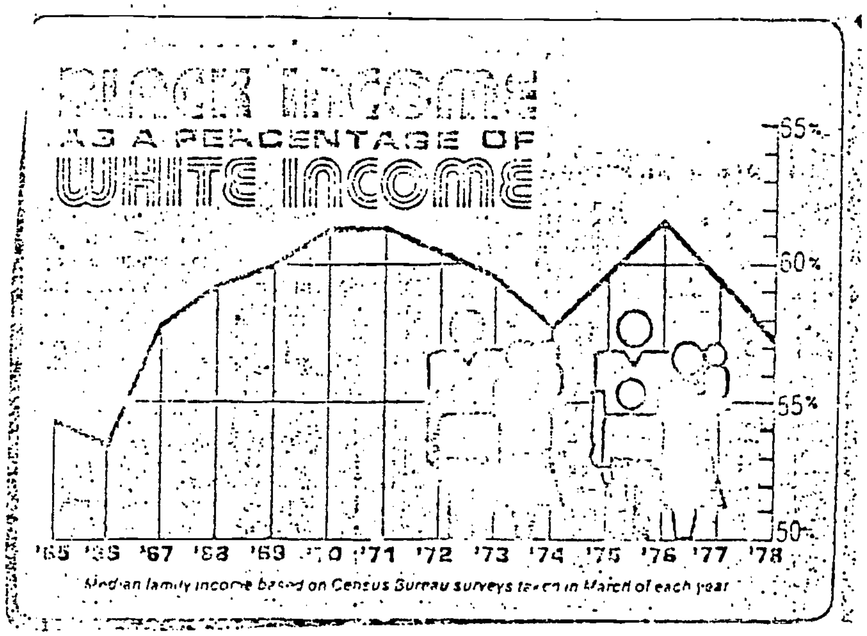
Transforming Use Class D: Consistency

Scales are used to accomplish consistency of reporting over time and place. Examples include the Dow-Jones industrial stock averages (DJIA), Consumer Price Index (CPI), IQ and SAT scores, and the various standardized test scales. In order to find a value on these scales, several pieces of data are needed (prices of several stocks, costs of a number of goods and services, responses to dozens of test items), and then this data is combined in some way to arrive at the scaled value. Often the ways in which these scales are defined must be modified from time to time so that the numbers can be given a consistent interpretation even though the outside world changes. For example, today's market basket for the CPI is different from the one used thirty years ago, and today's SAT exams use different items than those of just a few years ago.

We transform data to place it on these scales. So the ability to take advantage of a scale becomes a reason for transforming.

Examples:

1. The graph below shows median Black family income transformed into a percentage of median White family income. (a) If median White family income was about $14,300 in 1975, what was the median Black family income in that same year? (b) What is the advantage of transforming into these percentages? (Wall Street Journal, March 6, 1978)



Answers: (a) about $8500, substantially less than the median White family income. (b) Since both incomes change over time, the percentage enables consistent comparison.

Comment: Using the median incomes alone would not fairly show ups and downs in median Black family income, since inflation erodes much of what are generally upward trends in recent years. White income in each year is a better reference value.

Comment: The dips in 1966, 1974, and 1978 are the result of Blacks being harder hit in years of recession.

2. The chart below converts "number of items correct" into SAT scores. On these SAT tests, what are the scores of a student who responds to all items and correctly answers 80% of them on each part?

Score Conversion Table
Scholastic Aptitude Test
Form Code 8B210

Raw Score	College Board Reported Score		Raw Score	College Board Reported Score	
	SAT-Verbal	SAT-Math		SAT-Verbal	SAT-Math
85	800		40	470	600
84	790		39	460	590
83	780		38	450	590
82	770		37	450	580
81	760		36	440	570
80	760		35	430	560
79	750		34	430	550
78	740		33	420	540
77	730		32	410	540
76	720		31	410	530
75	720		30	400	520
74	710		29	390	510
73	700		28	390	500
72	690		27	380	490
71	690		26	370	490
70	680		25	370	480
69	670		24	360	470
68	660		23	350	460
67	650		22	350	450
66	650		21	340	440
65	640		20	330	430
64	630		19	330	430
63	620		18	320	420
62	610		17	310	410
61	610		16	310	400
60	600	800	15	300	390
59	590	780	14	290	380
58	590	770	13	290	380
57	580	760	12	280	370
56	570	750	11	270	360
55	570	740	10	270	350
54	560	730	9	260	340
53	550	720	8	250	330
52	550	710	7	250	330
51	540	700	6	240	320
50	530	690	5	230	310
49	530	680	4	220	300
48	520	670	3	220	290
47	510	660	2	210	280
46	510	650	1	200	280
45	500	640	0	200	270
44	490	640	−1	200	260
43	490	630	−2	200	250
42	480	620	−3	200	240
41	470	610	−4	200	230
			−5	200	220
			−6	200	220
			−7	200	210
			−8 or below	200	200

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Answer: 660 on Verbal, 670 on Math.

Comment: A different SAT test might have a different chart, but the tests are statistically equated so that the SAT scores are roughly equivalent regardless of which test is taken. This enables consistent interpretation of scores over a period of years.

3. A teacher customarily gives 100-point tests. At the end of the year a 40-item multiple choice standardized test is given. What can the teacher do to the raw scores on this test to make them equivalent to raw scores on the other tests?

Answer: Since 100/40 = 2.5, the raw scores on the standardized test can be multiplied by 2.5.

Comment: This still might not make the scores equivalent. If the standardized test is somewhat harder than the teacher's tests, then the scores on the standardized test will be lower and may have a wider range than the teacher's tests. As a result, the standardized test may assume greater importance even though it seems to be weighted equally. This is one reason why the various standardized tests are normalized and not merely weighted.

Summary

By *transforming* is meant the converting of numerical information into other numerical information that may have little or no resemblance to the original. The types of transformations considered in this chapter include ranking, shifting, scaling, log scaling, and standardizing.

A transformation may be required because of situational constraints. The resulting numbers may be easier to interpret (clarity). They may be easier to operate with, particularly if the given numbers are very small or very large and the results of transforming are small whole numbers (facility). Transformation may make it possible to compare numbers despite their arising out of different situations or from different times (consistency). Thus one transforms for the same reasons as one does other maneuvers.

Many transformations require a knowledge of mathematics beyond the scope of this volume. Though transformations are only briefly exemplified in this chapter, a thorough discussion would require considerably more space and would entail a great variety of examples.

Pedagogical Remarks

The arithmetic discussed in this chapter is not found in the standard curriculum, but is very closely related to standard ideas. Students can be asked to rank in discussions of the ordinal numbers first, second, third, etc. Percentiles can be taught as an application of percentages. Scaling is a wonderful application of division, and quickly involves multiplication, ratio, and proportion. Because transforming is usually performed for more than one of the use class reasons, we suggest focussing on the various types of transformations and letting all of the uses derive at once.

Ranking. Ranking can be tied to lessons in other areas. For instance, in history, one could discuss the changes in rank of the most populous states in the United States, and connect that with the move westward. In geography, one can rank the areas of the continents, or the lengths of the longest rivers in the world. In health, one could determine the calorie count of the breakfasts of students and rank them. In all these cases, the effect of ranking is to give a qualitative comparison from the quantitative (numerical) information. For instance, the population of North Carolina according to the 1790 census was 393,000. This number does not say as much as knowing that North Carolina ranked 3rd among the states at that time (Virginia was first and Pennsylvania second). Transforming populations into ranks tells us something about the United States at that time that the populations themselves do not.

Percentiles. It is distressing to realize that, on any measure, half of those things measured must fall below the 50th percentile.

Similarly, if grade level is measured as the median score of a large group on a test, half of the children in that group are below grade level. These notions are obvious from the mathematics of the situation, but they are difficult to translate into school policies: every school wants all of its children to be reading at grade level. A discussion of percentiles can be rather revealing in this regard. Grade levels are based upon percentiles and so can enter into the same kind of discussion.

Though sometimes used when a small collection of numbers are being ordered, percentiles are most meaningful when there are hundreds of numbers involved. For classroom examples, this is hard to do, and smaller collections may have to suffice.

How tall are you? is a possible question with which to begin. A child answers: "5 feet, 3 inches". The answer does not tell us whether this child is tall for his or her age. Take the heights of everyone in a class. Order them and calculate percentiles. The percentile of height that a person falls in may be a more meaningful number than the original heights.

Scaling. Scaling is an important transformation, and the Consumer Price Index is a particularly good example to use, because some salaries are based upon it. The idea is as follows. In a particular year, some items are priced. Say the total is \$357.23. This number is set equal to 100. Now if the same items cost \$450 some years later, we calculate the Consumer Price Index by solving the proportion

$$\frac{450}{357.23} = \frac{CPI}{100}$$

Here we would get a CPI of 126.0.

Ofttimes in tables or graphs, data will be given "in millions" or "in thousands" or "in 1979 dollars". This scales the data, for the distance from the Earth to the Sun in millions of kilometers is about 150, not 150,000,000. Despite the ubiquity of this kind of scaling in applications, problems requiring this scaling are seldom found in schoolbooks.

Standardizing. Students are often grouped into classes on the basis of test scores, yet they are seldom instructed regarding the meaning of those scores. Many students do not realize that a difference of one point in raw score may mean a difference of 5 to 10 percentiles. They may be surprised at how few or how many answers correct are needed in order to achieve a particular level.

Questions

1. (a) Rank the state (or province or country) in which you live against the other states of the U.S. (or provinces of Canada, or other countries on your continent) in terms of population and land area. (b) Give the percentile rank in each of these characteristics. (c) What information is given by these ranks that is not in the raw data?
2. Apply a transformation to average the following temperatures. (Different transformations may be appropriate for the different sets.)
   (a) 73°, 71°, 75°, 74°, 79° (high temperatures for a week)
   (b) -23°, -21°, -7°, -14°, -2° (low temperatures for a week)
3. From some source, determine the present value of the Consumer Price Index. Unleaded gasoline cost about $1.40 per gallon in 1981, when the CPI was about 260. (a) What should it cost at the time you are answering this question, if gasoline price is acting like other consumer prices? (b) How does this compare with the actual cost?
4. A student is tested in January of the 7th grade and found to be reading at the 11th grade level. (a) What does "reading at the 11th grade level" mean? (b) Since the student was not given an 11th grade test, how could such a determination be made? (c) Discuss the advantages and disadvantages of taking a raw score and converting it into a grade level equivalent.

5-11. The next three pages display conversion tables for the Metropolitan Achievement Tests Primary II (Harcourt Brace Jovanovich, 1971). Given are the raw scores of 20 students, named A through T, on Form G of the Math Computation subtest, taken at the end of Grade 2.

Student	Raw Score	Local Rank	Standard Score	Grade Equivalent	Stanine	National Percentile	Local Percentile
A	30						
B	11						
C	31						
D	29						
E	21						
F	31						
G	18						
H	33						
I	27						
J	20						
K	14						
L	24						
M	29						
N	26						
O	22						
P	31						
Q	17						
R	15						
S	25						
T	26						

(a) Complete each column of the table. (Assume local percentiles are calculated from this group alone.) (b) For each column, indicate why the information in that column might be useful.

Table 1. Raw Score to Standard Score Conversion Table

STANDARD SCORES

Raw Score	Word Knowledge F	G	H	Word Analysis F,G,H	Reading F	G	H	Total Reading F	G	H	Spelling F	G	H	Math Computation F	G	H	Math Concepts F,G,H	Math Prob. Solving F	G	H	Total Mathematics F	G	H	Raw Score
55								52	53	51											53	53	53	55
54								51	53	51											53	53	53	54
53								51	52	51											52	52	52	53
52								51	52	50											52	52	52	52
51								51	52	50											52	52	52	51
50								50	51	50											51	51	51	50
49								50	51	50											51	51	51	49
48								50	51	49											51	51	51	48
47								50	50	49											50	50	50	47
46								49	50	49											50	50	50	46
45								49	50	49											49	49	49	45
44					86	86	86	49	49	48											49	49	49	44
43					77	77	77	49	49	48											48	48	48	43
42					70	70	70	48	49	48											47	47	47	42
41					65	66	65	48	48	47											46	46	46	41
40	87	87	87		62	63	62	48	48	47							97				45	45	45	40
39	74	77	74		60	62	62	48	48	47							91				44	44	44	39
38	67	70	67		59	61	61	47	47	46							86				43	43	43	38
37	64	67	64		57	60	60	47	47	46							81				41	41	41	37
36	62	65	62		56	59	59	47	47	46							76				40	40	40	36
35	61	64	61	72	55	58	58	46	46	45							73	94	94	94	39	39	39	35
34	59	62	59	67	55	57	57	46	46	45							71	87	87	87	38	38	38	34
33	58	61	58	63	54	56	56	45	45	44				89	89	89	69	82	82	82	37	37	37	33
32	57	60	57	61	54	56	56	45	45	43				82	82	82	67	78	78	78	36	36	36	32
31	56	59	56	59	53	55	55	44	44	42				77	77	77	65	74	74	74	36	36	36	31
30	55	58	55	58	53	54	54	43	43	42	76	76	76	73	73	73	63	71	71	71	35	35	35	30
29	55	57	54	56	52	54	54	42	42	41	69	69	69	69	69	69	61	68	68	68	34	34	34	29
28	54	56	53	55	52	53	53	41	41	40	64	65	64	67	67	67	59	66	66	66	33	33	33	28
27	54	55	52	54	52	53	53	40	40	39	62	62	62	65	65	65	58	65	65	65	33	33	33	27
26	53	54	52	53	51	52	52	39	39	38	60	61	60	62	63	63	56	64	64	64	32	32	32	26
25	53	53	51	52	51	52	52	38	38	37	59	60	59	60	61	61	54	62	62	63	31	31	31	25
24	52	53	51	51	50	51	51	37	37	37	58	59	58	58	60	60	53	61	61	62	30	30	30	24
23	52	52	50	50	50	50	50	36	36	36	57	58	57	57	59	59	52	60	60	61	30	30	30	23
22	51	52	50	49	49	50	50	35	35	35	56	57	56	56	57	57	51	58	58	59	29	29	29	22
21	51	51	49	47	49	49	49	34	34	34	56	56	56	55	56	56	50	57	57	58	28	28	28	21
20	50	50	49	46	48	48	48	32	32	32	55	56	55	54	54	54	49	55	55	56	27	27	27	20
19	50	50	48	45	47	47	47	31	31	31	54	55	54	53	53	53	48	54	54	55	26	26	26	19
18	49	49	48	44	46	46	46	29	29	29	53	54	53	52	52	52	46	52	52	53	25	25	25	18
17	48	48	47	43	44	44	44	28	28	28	52	53	52	51	51	51	44	51	51	52	24	24	24	17
16	47	47	46	42	42	42	42	27	27	27	51	52	51	50	50	50	42	50	50	51	23	23	23	16
15	46	46	44	41	40	40	40	25	25	25	50	51	50	49	49	49	40	49	49	50	22	22	22	15
14	44	44	43	40	38	38	38	23	23	23	49	50	49	48	48	48	38	47	47	48	21	21	21	14
13	43	43	42	38	36	36	36	21	21	21	49	49	49	46	46	46	36	45	45	47	20	20	20	13
12	42	42	41	37	34	34	34	19	19	19	48	49	48	45	45	45	34	43	43	45	20	20	20	12
11	41	41	39	35	32	32	32	17	17	17	47	48	47	43	43	43	32	41	41	43	19	19	19	11
10	39	39	37	33	30	30	30	16	16	16	46	47	46	40	40	40	31	38	38	39	18	18	18	10
9	37	37	35	31	28	28	28	14	14	14	45	46	45	37	37	37	29	35	35	36	17	17	17	9
8	35	35	33	29	26	26	26	13	13	13	44	45	44	35	35	35	28	33	33	33	15	15	15	8
7	33	33	31	26	24	24	24	11	11	11	43	44	43	33	33	33	27	30	30	30	14	14	14	7
6	31	31	30	24	22	22	22	9	9	9	41	43	41	32	32	32	26	28	28	28	13	13	13	6
5	29	29	28	21	20	20	20	8	8	8	40	41	40	30	30	30	25	26	26	26	11	11	11	5
4	26	26	26	19	18	18	18	6	6	6	39	40	39	28	28	28	24	25	25	25	10	10	10	4
3	24	24	24	17	15	15	15	5	5	5	38	39	38	27	27	27	24	23	23	23	9	9	9	3
2	20	20	20	15	13	13	13	4	4	4	36	37	36	26	26	26	23	22	22	22	7	7	7	2
1	14	14	14	12	9	9	9	2	2	2	35	35	35	25	25	25	23	21	21	21	6	6	6	1

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Table 3. Standard Score to Percentile Rank and Stanine Conversion Table

For Beginning of Grade 2 (Primary II)

STANDARD SCORES

Stanine	%ile Rank	Word Knowledge	Word Analysis	Reading	Total Reading	Spelling	Math Computation	Math Concepts	Math Prob. Solv.	Total Math	%ile Rank	Stanine
	99	73	70	74	80	69	67	71	68	83	99	
9	98	70	67	71	74	65	62	69	66	78	98	9
	96	67	64	69	63	64	58	65	65	71	96	
	94	64	62	66	62	62	57	63	64	67	94	
8	92	62	60	64	61	61	56	61	62	64	92	8
	90		58	62				59	61	63	90	
	89	61			60	60	55				89	
	88	60	57	61	59			58	60	62	88	
	86	59	56	60	58	59	54	56	58	60	86	
	84	58	55	59	57	58	53	55		58	84	
7	82	57	54	58	56			54	57	57	82	7
	80	56	53	57	55	57	52		56		80	
	78	55	52	56	54	56		53	55	56	78	
	77										77	
	76	54	51	55	53	55	51		54	55	76	
	74	53		54	52	54		52	53		74	
	72	52	50	53	51	53			52	54	72	
	70		49	52	50	52	50	51			70	
6	68		48	51	49	51			51	53	68	6
	66	51				50	49	50			66	
	64		47	50	48				50	52	64	
	62	50	46	49		49		49			62	
	60	49			47		48		49	51	60	
	58	48		48		48		48	48		58	
	56	47	45		46		47			50	56	
	54			47		47		47	47		54	
	52						46		46	49	52	
5	50	46	44	46	45	46		46	45	48	50	5
	48	45		45			45	45	44		48	
	46		43	44	44	45	44	44	43	47	46	
	44	44						43	42		44	
	42		42	43	43		43			46	42	
	40	43				44	42	42	41	45	40	
	38		41	42	42		41	41	40	44	38	
	36	42				43	40	40	39	43	36	
	34		40	41	41		39	39	38	42	34	
	32	41	39			42	38		37	41	32	
4	30			40	40		37	38	36	40	30	4
	28	40	38	39		41	36	37	35	39	28	
	26				39				34	38	26	
	24	39	37	38	38		35	36	33		24	
	23								32		23	
	22	38		37		40	34	35	31	37	22	
	20	37	36	36	37		33	34	30	36	20	
	18		35	35	36			33	29		18	
3	16	36			35	39		32	28	35	16	3
	14	35	34	34	34	38	32		27	34	14	
	12	34	33		33		31	31		33	12	
	11	33		33		37		30	26		11	
	10	32	32	32	32		30	29		32	10	
2	8	31	31	31	30	36	29		25	31	8	2
	6	29	30	29	29	35	28	28	24	30	6	
	4	27	29	28	28		27	27	23	28	4	
1	2	26	27	27	27		26	26	22	26	2	
	[illegible]	22	22	23	23		25	25	21	21	[illegible]	

Table 3: Standard Score to Percentile Rank and Stanine Conversion Table

For End of Grade 2

STANDARD SCORES

Stanine	%ile Rank	Word Knowledge	Word Analysis	Reading	Total Reading	Spelling	Math Computation	Math Concepts	Math Prob. Solv.	Total Math	%ile Rank	Stanine
	99	85	72	83	86	77	82	91	94	88	99	
9	98	79	70	80	81	76	77	86	87	84	98	9
	96	76	69	77	75	75	73	81	82	81	96	
	94	74	68	73	70	74	69	76	78	78	94	
8	92	70	67	70	69		68	73	74	75	92	8
	90	67	65	68	68	73	67	72	73	73	90	
	89	66	64	67	66				72		89	
	88	65		66	65	72	66	71	71	72	88	
	86	64	63	64	64	71	65	70	68	70	86	
	84	63	62	63	63	70	63	69	67	69	84	
7	82			62	62	69	62		66	67	82	7
	80	62	61	61	61	68	61	68		66	80	
	78		60		60	67	60	67	65	65	78	
	77			60		66					77	
	76	61	59			65	59	66		64	76	
	74	60			59	64	58	65	64		74	
	72	59		59	58	63			63	63	72	
	70		58	58	57	62		64		62	70	
6	68	58	57	57		61	57	63	62		68	6
	66		56		56					61	66	
	64	57		56		60		62	61		64	
	62		55		55		56	61		60	62	
	60			55		59		60	60		60	
	58	56	54				55	59	59	59	58	
	56	55			54	58					56	
	54		53	54					58	58	54	
	52				53	57	54	58			52	
5	50	54	52	53				57	57	57	50	5
	48				52	56		56			48	
	46		51				53		56		46	
	44	53	50	52	51			55		56	44	
	42					55		54	55		42	
	40			51			52				40	
	38	52	49		50	54		53	54	55	38	
	36		47	50		53	51		53		36	
	34				49	52			52	54	34	
	32	51	46	49				52			32	
4	30					51	50		51		30	4
	28		45	48	48	50		51		53	28	
	26	50	44	47			49		50		26	
	24					49		50		52	24	
	23			46	47						23	
	22	49	43	45			48	49	49		22	
	20	48	42	44	46	48	47			51	20	
	18			42		47	46	48	48		18	
3	16	47	41	41	45	46		47	47	50	16	3
	14	46	40	40	44	45	45	46	45	49	14	
	12	45	39	39	43	44	44	44	44	48	12	
	11	44		38	42	43	43	43		47	11	
	10	43	38	37	41	41	42	42	43	46	10	
2	8	42	37	36	40	40	40	40	41	41	8	2
	6	41	35	35	38	38	37	38	38	41	6	
	4	39	34	34	37	36	35	36	35	39	4	
1	2	37	33	32	35	35	33	34	33	36	2	
	[illegible]	31	26	25	29		28	31	[illegible]	[illegible]	[illegible]	

Table 1. Raw Score to Standard Score Conversion Table (continued)

STANDARD SCORES

Raw Score	Total Reading F	Total Reading G	Total Reading H	Total Mathematics F	Total Mathematics G	Total Mathematics H	Raw Score
108				102	102	102	108
107				99	99	99	107
106				96	96	96	106
105				94	94	94	105
104				92	92	92	104
103				90	90	90	103
102				88	88	88	102
101				87	87	87	101
100				85	85	85	100
99				83	83	83	99
98				81	81	81	98
97				80	80	80	97
96				79	79	79	96
95				78	78	78	95
94				76	76	76	94
93				75	75	75	93
92				74	74	74	92
91				73	73	73	91
90				72	72	72	90
89				71	71	71	89
88				70	70	70	88
87				69	69	69	87
86				68	68	68	86
85				67	68	68	85
84	94	94	94	66	67	67	84
83	86	86	86	65	67	67	83
82	81	81	81	65	66	66	82
81	75	75	75	64	66	66	81
80	70	70	70	63	65	65	80
79	68	68	68	63	65	65	79
78	65	66	65	62	64	64	78
77	63	64	63	62	64	64	77
76	62	63	62	61	63	63	76
75	61	62	61	61	63	63	75
74	60	61	61	60	62	62	74
73	59	61	60	60	62	62	73
72	58	60	59	59	61	61	72
71	57	60	59	59	61	61	71
70	57	59	58	58	60	60	70
69	56	59	58	58	60	60	69
68	56	58	57	57	59	59	68
67	55	58	57	57	58	58	67
66	55	57	56	57	58	58	66
65	55	57	56	56	57	57	65
64	54	57	55	56	57	57	64
63	54	56	55	56	56	56	63
62	54	56	54	55	56	56	62
61	53	55	54	55	56	56	61
60	53	55	53	55	55	55	60
59	53	54	53	54	55	55	59
58	52	54	52	54	54	54	58
57	52	54	52	54	54	54	57
56	52	53	52	53	54	54	56

Table 2. Standard Score to Grade Equivalent (G.E.) Conversion Table

STANDARD SCORES

G.E.	Word Knowledge	Word Analysis	Reading	Total Reading	Spelling	Math Comput.	Math Concepts	Math Prob. Solv.	Total Math	G.E.
9.9	130		135	135	135	140	136	136	[illegible]	9.9
9.8				99	101			105	[illegible]	9.8
9.7	98						102			9.7
9.6				98		105		105		9.6
9.5			98		100		101	104	110	9.5
9.4	97									9.4
9.3	96		97	97	99		100			9.3
9.2						104			109	9.2
9.1	95				98		99	103		9.1
9.0			96	96					108	9.0
8.9					97		98			8.9
8.8	94					103		102	107	8.8
8.7			95	95	96		97			8.7
8.6	93								106	8.6
8.5						102	96	101		8.5
8.4	92		94	94					105	8.4
8.3					95	101	95	100		8.3
8.2			93	93					104	8.2
8.1	91				94	100	94	99	103	8.1
8.0			92	92					102	8.0
7.9	90				93	99	93	98		7.9
7.8			91	91					101	7.8
7.7	89				92	98	92	97		7.7
7.6			90	90					100	7.6
7.5	88		89		91	97	91	96	99	7.5
7.4				89				95		7.4
7.3	87		88	88	90	96	90	94	98	7.3
7.2	86				89					7.2
7.1	85		87	87		95	89	93	97	7.1
7.0					88			92		7.0
6.9	84		86	86		94	88		96	6.9
6.8					87	93		91	95	6.8
6.7	83		85	85	86	92	87		94	6.7
6.6				84		91		90	93	6.6
6.5	82		84		85	90	86	89		6.5
6.4				83	84			88	92	6.4
6.3	81		83			89	85	87		6.3
6.2				82	83	88			91	6.2
6.1	80		82		82	87	84	36		6.1
6.0	79		81	81	81	86		85	90	6.0
5.9			80			85	83		89	5.9
5.8	78			80	80			84		5.8
5.7			79	79		84	82		88	5.7
5.6	77			78	79	83	81	83	87	5.6
5.5		72	78	77		82	80	82		5.5
5.4	76		77		78	81		81	86	5.4
5.3	75		76	76		80	79		85	5.3
5.2	74			75	77		78	80	84	5.2
5.1			75		76	79	77	79	83	5.1
5.0	73			74	75	78				5.0
4.9			74	73	74	77	76	78	82	4.9
4.8	72		73	72	73	76	75		81	4.8
4.7	71	67	72		72	75	74	77	80	4.7
4.6				71	71	74	73	76	79	4.6
4.5	70		71	70		73		75	78	4.5
4.4	69		70	69	70	72	72	74	77	4.4
4.3		64	69	68	69	71		73	76	4.3
4.2	68		68	67	68	69	71		75	4.2
4.1	67	63	67	66	67	68	70	72	74	4.1
4.0						67	69	71	73	4.0
3.9	66	62	66	65	66	66	68	70	72	3.9
3.8	65	61	65	64		65	67	69	71	3.8
3.7	64	60	64	63	65	64	66	58	70	3.7
3.6	63	59	63	62	64	63	65	67	69	3.6
3.5	62		62	61	63	62	64	56	67	3.5
3.4	61	58	61	60	62	61	63	65	66	3.4
3.3	60	57	60	59	61	60	62	64	65	3.3
3.2	59	56	59	58		59	60	62	63	3.2
3.1	58	55	58	57	60	57	59	61	62	3.1
3.0	57	54	57	56	59	56	58	59	60	3.0
2.9	56	53	56	55	57	55	56	57	[illegible]	2.9
2.8	55	52	55	54	56	53	55	55	58	2.8
2.7	54	51	54	53	55	52	53	54	57	2.7
2.6	53	50	52	52	54	51	52	52	56	2.6
2.5	52	49	51	51	52	50	51	51	55	2.5
2.4	50	47	50	49	51	49	50	50	53	2.4
2.3	49	46	49	48	49	48	49	[illegible]	[illegible]	2.3
2.2	47	45	47	46	47	46	[illegible]	[illegible]	[illegible]	2.2
2.1	46	44	46	45	46	45	[illegible]	[illegible]	[illegible]	2.1
2.0	45	42	45	43	45	43	[illegible]	[illegible]	[illegible]	2.0
1.9	42	40	43	41	43	[illegible]	[illegible]	[illegible]	[illegible]	1.9
1.8	40	38	41	39	42	39	[illegible]	[illegible]	[illegible]	1.8
1.7	38	37	39	37	41	37	[illegible]	[illegible]	[illegible]	1.7
1.6	36	35	37	35	40	36	[illegible]	[illegible]	[illegible]	1.6
1.5	34	34	[illegible]	32	39	35	35	[illegible]	[illegible]	1.5
1.4	31	33	[illegible]	29	37	34	[illegible]	[illegible]	[illegible]	1.4
1.3	27	31	29	26	35	32	[illegible]	[illegible]	[illegible]	1.3
1.2	24	29	[illegible]	[illegible]		31	[illegible]	[illegible]	[illegible]	1.2
1.1	[illegible]	26	[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	1.1
1.0	18	[illegible]	[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	1.0

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Notes and Commentary

1. Types of transformations
2. Alternate organizations
3. Process vs. Outcome
4. Alias vs. Alibi

1. Types of transformations. There are numerous types of transformations, including types not mentioned in this chapter. A common way to classify them is by their preservation properties. In this regard,

(a) ranking preserves order only,
(b) shifting preserves distance between numbers (and thus many other things based upon distance, such as standard deviation),
(c) scaling preserves ratios of numbers.

The results of transformations like this are called scales, as discussed in Chapter 1, Section D, even though technically scaling is the creation of a new measure. Some statisticians distinguish four kinds of scales: nominal or classificatory scales (e.g., numbers on football jerseys) that have few arithmetic properties, ordinal or ranking scales, interval scales, and ratio scales.

Exponential scales (e.g., the Richter scale, star magnitudes, and the decibel scales mentioned in Chapter 1, Section D) convert equal ratios into equal distances. They are not exemplified in this chapter because the mathematics is beyond the scope of this book. Normalized scales convert data into relative positions on the bell-shaped curve.

Other types of transformations can be more complicated. For instance, the approximation of data by a line of best fit converts given data into new data in a way that does not preserve distance or ratio, but the new data can be described by a simple formula and the sum of the squares of the changes in values is minimized.

2. Alternate organizations. The above discussion suggests that we might have organized this chapter around the various types of transformations. This alternative was considered and not followed because it would have necessitated a discussion in more depth than the importance of transformations in the present context warranted, and because many statistics books discuss the uses and benefits of various transformations of data.

We opted for consistency among the Chapters of Part III. This required that we fit the examples into the four use classes. Often a single example fits into many use classes, and even our final sorting may not sit well with some readers. For instance, a major use of the Consumer Price Index is to equate prices at different times despite inflation, and thus to provide a means to consistency, whereas we used it to exemplify situational necessity. The CPI also facilitates comparison of prices in various eras. Many of the examples straddle use classes in similar fashion.

3. Process vs. Outcome. Many people have difficulty distinguishing scales from scaling or transforming. The process of transforming leads to the outcome called a scale. Because the two ideas are so inherently related, any classification of transforming suggests a classification of scales. (Analogously, the various processes of estimating give rise to various kinds of estimates.)

Our discussion of scales as such is found under the uses of number, in Chapter 1, Section D. Numbers on scales are locations. This chapter is devoted to answering why one would want to take numerical information and transform it to a scale.

4. Alias vs. Alibi. Is a change of unit (which we classified under rewriting) a transformation? Is equating for changes in time, as in dividing by the CPI, only sophisticated rewriting as with a unit of "constant dollars"? The answer is that either interpretation is possible. When something has been rewritten (to its alias), we can always judge the rewriting process to be a transformation. When something has been placed in a different context (the alibi), as on a different scale, we can always say that we have the same attribute (a score or value) but are denoting it differently.

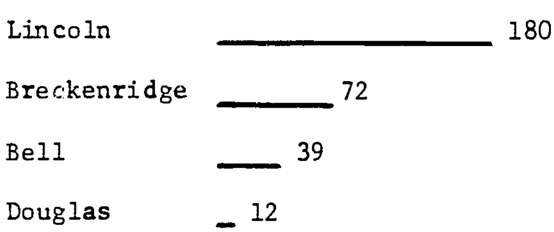
A good example of choice in interpretation is the conversion from Fahrenheit to Celsius temperatures. The conversion could be considered as a transformation with the formula $C = (5/9)(F - 32)$ or as merely rewriting in a different system. To some extent all maneuvers, including transforming, displaying, and estimating, are sophisticated forms of rewriting.

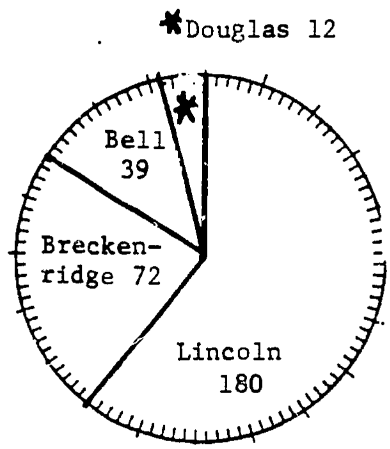
CHAPTER 14

REASONS FOR DISPLAYING

A <u>display</u> of numerical information is a visual presentation of that information. Displays may take on a variety of forms, including lists, tables, charts, diagrams, graphs, maps, and prose. Below are four displays of the results of the 1860 U.S. presidential election: prose, bar graph, circle graph, and table. Still other displays are possible with this data. For example, one often sees maps of the United States showing election results on a state-by-state basis.

> In the 1860 presidential election, Abraham Lincoln received 180 electoral votes, John C. Breckenridge 72 electoral votes, John Bell 39 electoral votes, and Stephen A. Douglas, 12 electoral votes.





Electoral Votes--1860 Election

Major Parties' Popular and Electoral Vote for President

(F) Federalist; (D) Democrat; (R) Republican; (DR) Democrat Republican; (NR) National Republican; (W) Whig; (P) People's; (PR) Progressive; (SR) States' Rights; (LR) Liberal Republican; Asterisk (*)—See notes.

Year	President elected	Popular	Elec.	Losing candidate	Popular	Elec.
1789	George Washington (F)	Unknown	69	No opposition		
1792	George Washington (F)	Unknown	132	No opposition		
1796	John Adams (F).	Unknown	71	Thomas Jefferson (DR)	Unknown	68
1800*	Thomas Jefferson (DR)	Unknown	73	Aaron Burr (DR)	Unknown	73
1804	Thomas Jefferson (DR)	Unknown	162	Charles Pinckney (F)	Unknown	14
1808	James Madison (DR).	Unknown	122	Charles Pinckney (F)	Unknown	47
			. . .			
1856	James C. Buchanan (D)	1,927,995	174	John C. Fremont (R).	1,391,555	114
1860	Abraham Lincoln (R)	1,866,352	180	Stephen A. Douglas (D)	1,375,157	12
				John C. Breckinridge (D)	845,763	72
				John Bell (Const. Union).	589,581	39
1864	Abraham Lincoln (R)	2,219,067	212	George McClellan (D)	1,808,725	21
1868	Ulysses S. Grant (R)	3,015,071	214	Horatio Seymour (D).	2,709,615	80
1872*	Ulysses S. Grant (R)	3,597,070	286	Horace Greeley (D-LR)	2,834,079	. . .

From <u>The World Almanac</u>, 1980, p. 279

The person who generates data may have a choice of display. However, it is more common to encounter information already displayed than to generate it. It is obvious that there are many ways to display the same information. In this chapter we analyze the reasons for choosing particular kinds of displays over others.

The reasons for these choices can be categorized under four headings, identical to those in Chapters 11 - 13.

A. Constraints (e.g., log-log graphs when data is of many different orders of magnitude)

B. Clarity (e.g., graphing to illuminate a relationship)

C. Facility (e.g., tabulating to enable easier comparison)

D. Consistency (e.g., placing dissimilar information on the same types of graphs)

Displaying Use Class A: Constraints

Unlike rewriting, estimating, or transforming, maneuvers which are often optional, displaying is necessary for all communications of numerical information (and other types of information as well) for which one wants a record. Thus we do not consider in detail the question "When does one display?" because the answer is "almost always".

However, the choice of display is subject to many constraints. One set of constraints deals with the graphic and type font capabilities of the person or machine doing the displaying. A second set of constraints is imposed by the numerical information itself, and usually revolves around space or perceptual limitations. Tables are usually designed to minimize the space the numbers occupy. Graphs are needed if there is too much information to tabulate. Special axes are needed on graphs if the numerical information involves a wide range of values. Prose is seldom used when there are many numbers involved because it takes too much space.

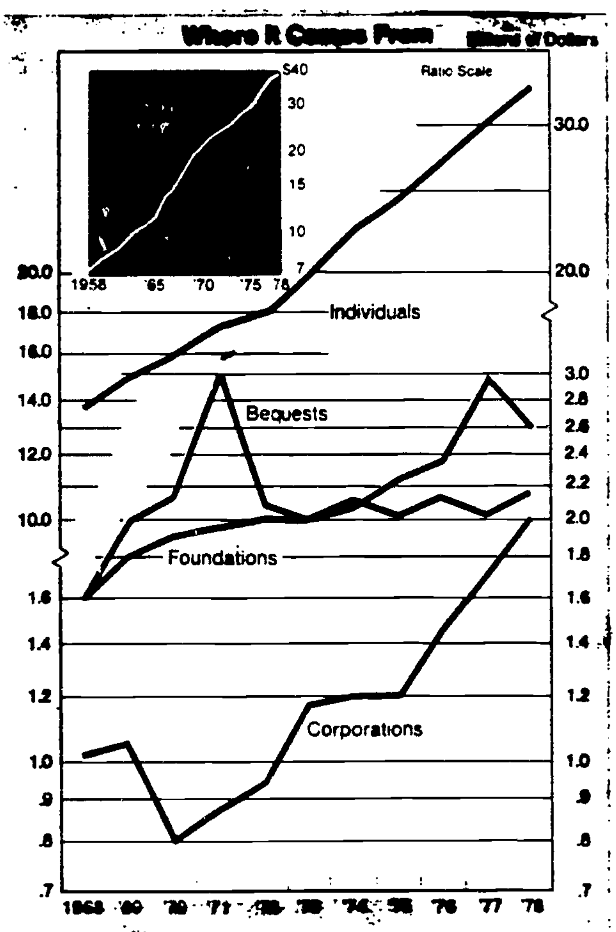
Examples:

1. Five separate items are graphed in the following chart. What are these five items and what are their values for the year 1975? (Source: The Conference Board.)

U.S. Philanthropy

Economic Road Maps
Nos. 1870-1871

January, 1980

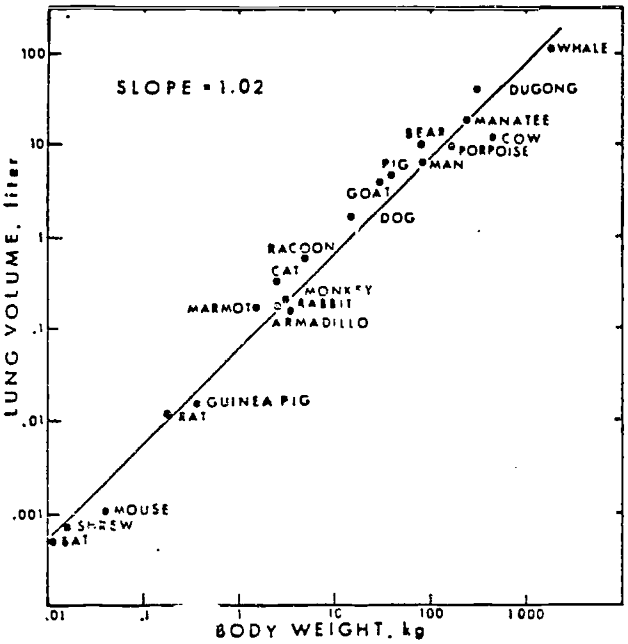


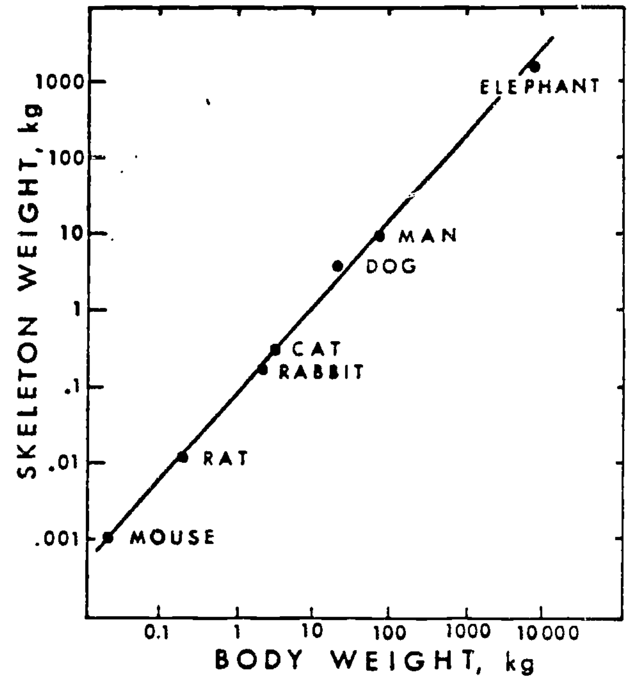
Answer: In 1975, approximately 29 billion dollars was contributed in the United States. Of this total, 1.2 billion dollars was contributed by corporations, 2.0 billion dollars by foundations, 2.3 billion dollars by bequests, and about 23.5 billion dollars by individuals.

Comment: The vertical scale is a ratio or logarithmic scale, utilized to accommodate all of the giving levels on the same graph. (Graph paper with a ratio scale on one axis is called semilog graph paper.) Even this scale does not allow all of the data to fit, so there are different breaks in both the left and right scales to accommodate the amounts of individual contributions. The complexity of this chart is rather unusual. Clarity has been sacrificed because of the constraints of space.

Comment: The small inset chart also utilizes a ratio scale. Such scales are helpful in prediction, for when something grows exponentially (as does compound interest), when graphed on this scale the graph will resemble a straight line. Thus we see that total contributions and individual contributions have grown nearly exponentially over the time interval considered.

2. Give the lung volume and skeleton weight for a man, a mouse, and a cat, as shown on these graphs. Source: Knut Schmidt-Nielsen, <u>How animals work</u> (Cambridge University Press, 1972), p. 89, p. 102.





Answers: A man has a lung volume of about 7 liters, skeleton weight of about 10 kg; for the mouse, about .001 liters and .001 kg; for the cat, about .4 liters and .6 kg.

Comment: Lung volume and skeleton weight are both multiples of powers of body weight. Here, if B = body weight, L = lung volume, and S = skeleton weight, then $L \approx .08B^{1.02}$ and $S \approx .07B^{1.1}$. The use of logarithmic scales on both axes, as done here, converts a power relationship into a linear one. It is quite difficult to read values from the graph when they are not integer powers of 10, but the purpose of the graphs is to show how closely the animals fit the relationship.

Comment: Graph paper with ratio scales on both axes is called log-log paper, particularly useful in situations like this, where both variables range widely.

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3. "Red ink" or "in the red" as phrases to describe business losses have their origin in the fact that in times past business ledger displays often indicated losses or debits by using red ink. Today that is not usually so; displays of financial data use signals other than color to indicate losses or debits. (a) What are some such signals? (b) Why has the use of color generally been discontinued?

Answer: (a) To indicate a loss of, let's say $150, used are -$150, ($150), or $150 (loss), or the word "loss" is displayed in a heading. (b) The main reason for discontinuing color has been that nearly all copying machines produce black and white copies no matter what is on the original and that printers on computers rarely have ways of printing in two colors.

Comment: Copying machines and computers have revolutionized business and accounting practices in the past few decades, and forced many customs dealing with presentations of data to be changed.

4. (a) In the table below, what do the numbers in the row beginning 1950 denote? (b) Why aren't the actual numbers used? Source: <u>Information Please Almanac</u>, 1983, p. 65.

Women in the Working Population

Year[1]	Number (thousands)	Percent of female population aged 10 and over[1]	Percent of total working population aged 10 and over[1]
1890	4,006	17.4	17.2
1900	5,319	18.8	18.3
1910	7,445	21.5	19.9
1920	8,637	21.4	20.4
1930	10,752	22.0	22.0
1940	12,845	25.4	24.3
1950	18,412	33.9	17.3
1960[2]	23,272	37.8	19.4
1970	31,583	43.4	22.5
1979	44,375	51.0	26.6
1980	45,646	51.6	26.9
1981	46,873	52.2	27.7

1. For 1880–1930, data relate to population and gainful workers at ages 10 and over; for 1940, to ages 14 and over; for 1950–78, to population at ages 16 and over. 2. Beginning in 1960, figures include Alaska and Hawaii. *Sources:* Department of Commerce, Bureau of the Census, and Department of Labor, Bureau of Labor Statistics.

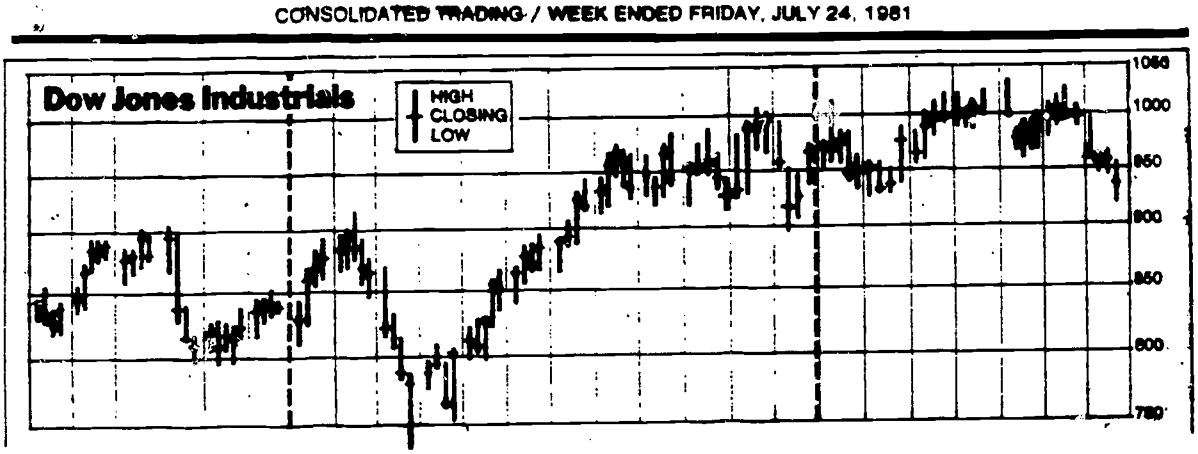
Answer: (a) In 1950, there were 18,412,000 women in the working population, equal to 33.9% of the female population and 17.3% of the total working population age 16 or older. (b) To save space and avoid unnecessary repetition.

Comment: It is very common in tables or graphs to see headings like the "Number (thousands)" heading used here. Such headings are seldom found in the corresponding exercises in schoolbooks.

Comment: The prose following the table is necessary in order to understand what the data signify.

5. How many pieces of information are given in the graph below? (Source: The Wall Street Journal, July 27, 1981.)

CONSOLIDATED TRADING / WEEK ENDED FRIDAY, JULY 24, 1981



Answer: The graph gives high, low and closing prices for 108 weeks and the corresponding months. The shape of the graph exhibits the range for each week and trends over the 2-year period. So there are at least 300 to 400 pieces of information.

Comment: To put so much information in so little space requires skillful charting techniques.

Displaying Use Class B: Clarity

Just as photographs taken from different angles can convey different aspects of an object or event, various types of displays clarify different aspects about numerical data.

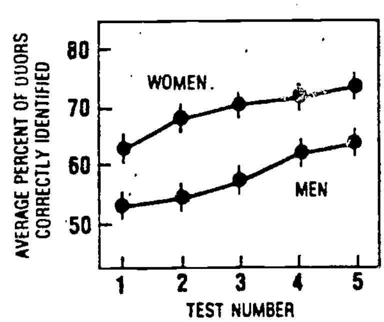
The well-known aphorism "A picture is worth a thousand words" applies to the ability of a well-constructed graph to exhibit relationships among numbers. Prose displays enable emphasis of qualitative aspects of data. Tables convey a large amount of sorted information in a brief space.

For purposes of clarity, two or more types of displays can be combined, thus taking advantage of the powers of each type. For instance, many diagrams and charts combine aspects of graphs and tables, and books often elaborate upon graphs with prose.

Examples:

1. Name the two types of displays used in this report and indicate what information is clarified by each.

WOMEN'S KEENER SENSE OF SMELL



Outcome of an experiment that tested 22 men and 24 women on learning 80 odors, over a period of 10 days. After a trial run, subjects were told the names of some of the odors. The ability of both men and women to come up with the correct names improved over time. But on the average, women correctly identified more items on the initial test and maintained their advantage in four later trials.

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Answer: This report combines a (Cartesian) coordinate graph with prose. The graph clarifies the improvement of both women and men over time to distinguish odors. The prose gives background data that is not so appropriate for graphing and also interprets two aspects of the graph that the authors feel are most important.

Comment: Students are often instructed to have axes intersect at (0, 0). In this case, such a constraint would either waste space on a much larger display or cause the men's and women's data to be too close to be clearly read.

Comment: A good simple question for young students is to pick a point (say the point at the upper right) and ask what it stands for. Answer: On the 5th test, women were able to identify 72% of the odors they had previously been instructed upon.

2. Examine the four presentations of the data for the 1860 presidential election as given at the beginning of the chapter. Which display clarifies best: (a) the ratios of electoral votes for the candidates; (b) the ratio of the votes for a given candidate with the total electoral vote; (c) how this election stacked up with other elections in the same era.

Answers: (a) the bar graph; (b) the circle graph; (c) the table.

Comment: When bar graphs start from 0, as this one does, the lengths of the bars convey a good feel for the ratios between the numbers. Circle graphs are best for comparing parts with a whole. Tables enable so much more information to be fit into a given space that descriptive information (here the parties of the candidates and who won or lost) and other numerical information (here the vote totals) can be included without reducing clarity.

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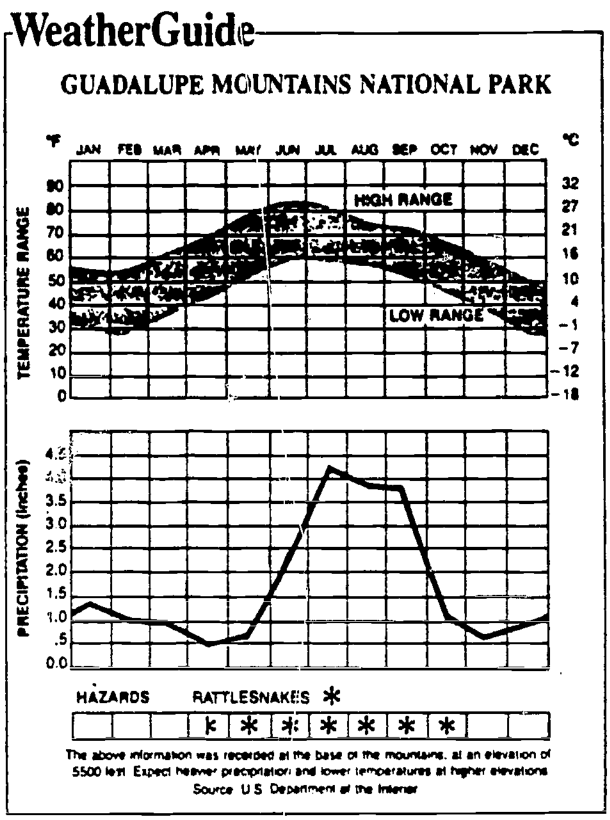
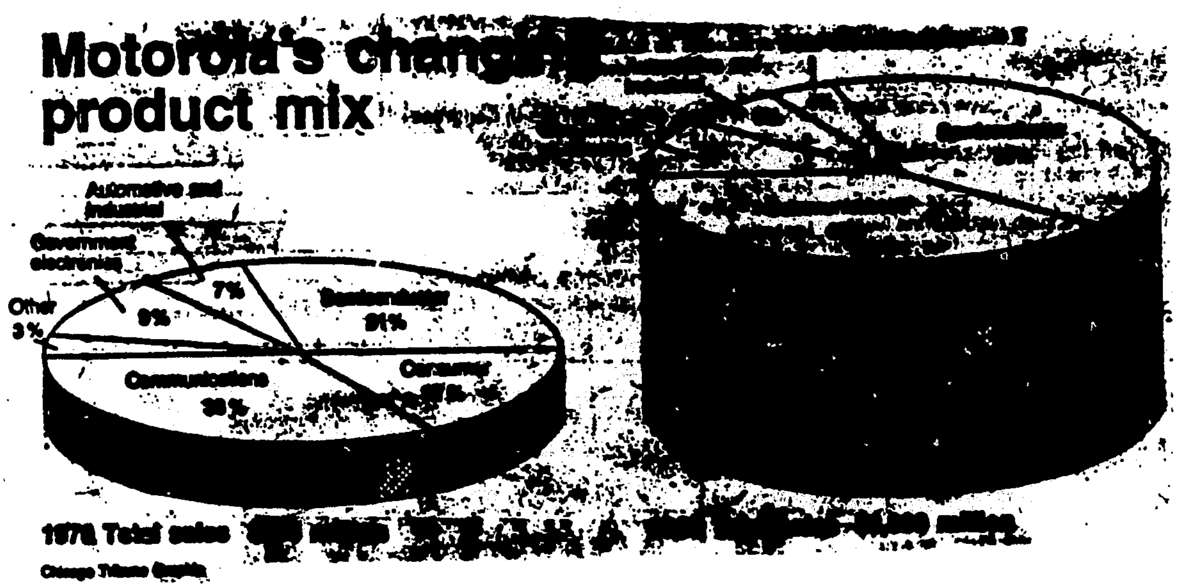
3. Here are two graphs dealing with the weather at Guadalupe Mountains National Park in western Texas. Name one feature of the weather emphasized in each graph. (Source: The Backpacker, June-July, 1981.)



Answers: The top graph emphasizes the range of temperatures each day (about 20°F or 12°C). The bottom graph indicates that the rainy season in the park is from June to September with very little rain at other times. (Other answers are possible.)

Comment: Each graph contains at least one confusing notion. At the top, since the park contains mountains, the word "range" has a dual meaning. Perhaps the park has two mountain ranges, one high and one low, with different temperatures at each. The shading is a clue that this is not meant, but it is a subtle clue. The confusion in the bottom graph can be seen by trying to determine the precipitation in June.

4. (a) What two types of displays are combined in this report from the Chicago Tribune, July 26, 1981? (b) What is clarified by this combination? (c) What possible deception is caused by this combination?



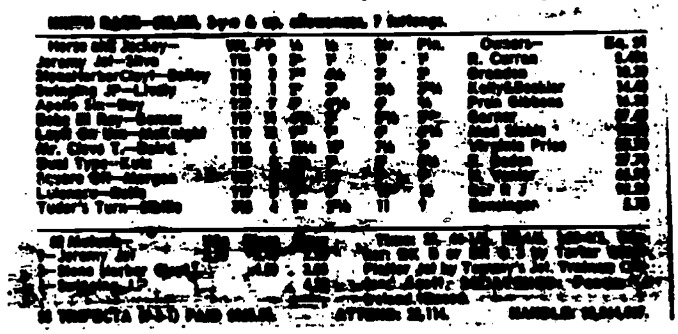
Answers: (a) A circle graph indicating the product mix is combined with a bar graph indicating sales for 1970 and 1980.

(b) The combination enables one to see the changing product mix at the same time as the total sales are described. The biggest change is in the consumer end.

(c) The bar graph is indicated by a three-dimensional picture. Though the height of the cylinder at right is (correctly) about four times that of the cylinder at left, the volume is about sixteen times. Thus the viewer is given an impression that total sales have grown much more than they did. (The visual impact is somewhere between four and sixteen times greater.)

Comment: One should further temper the 1980 total sales figure by adjusting for changes in the value of a dollar.

5. Below is the description of a horse race at Arlington Park, July 18, 1981, as given in the Chicago Sun-Times the next day. Convert the information for the line starting "Apollo Six-Day" into prose.



Answer: Apollo Six, ridden by jockey Day, carried a weight of 119 lbs. He started from post position 7 and after 1/4 mile was 8th by 3 lengths over the 9th place horse. After 1/2 mile he was 6th by 1 1/2 lengths. In the stretch he was 4th, where he finished 3/4 of a length ahead of the 5th place horse. Owned by Fran Gibbons, he would have paid $16.30 for each dollar bet had he won.

Comment: This question is easy for someone familiar with horse racing and racing charts, and difficult for others. Clarity depends upon one's familiarity with the context.

Comment: Imagine how many lines it would take to present by prose all of the information contained in the table, and how unclear that might be.

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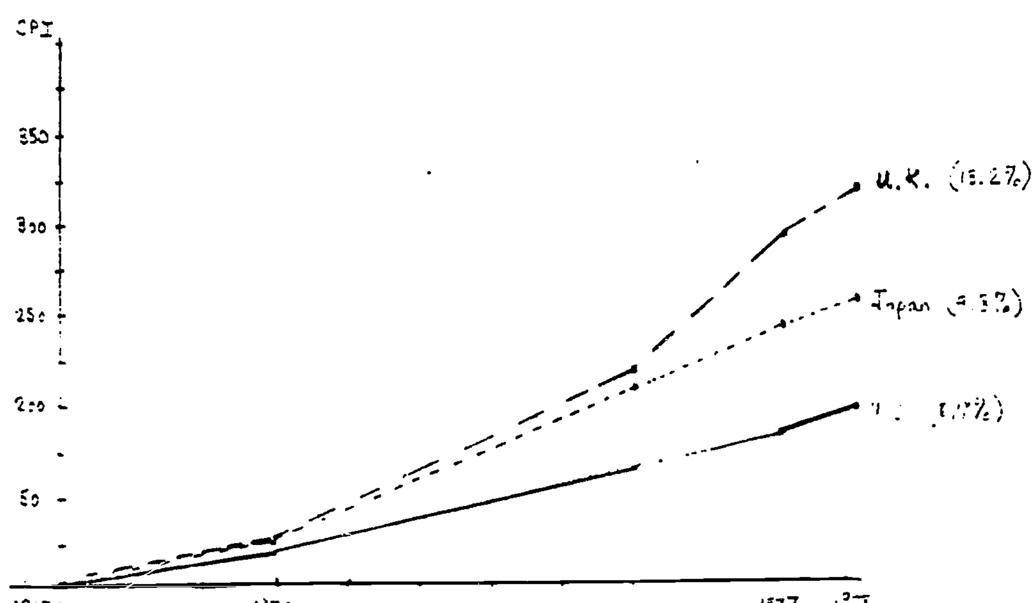
6. (a) From the following table of Consumer Price Indexes for Selected Countries, taken from the 1981 Information Please Almanac, p. 134, graph the total indexes for Japan, the United Kingdom, and the United States for the indicated years. (b) What does the graph exhibit that is not as clearly shown by the table?

Consumer Price Indexes for Selected Countries
(1967—100)

Country	Total Indexes 1978	1977	1975	1970	Average annual percent change, 1970–1978	Indexes for Selected Items, 1978: Food[1]	Clothing	Housing[2]	Transportation
Australia	245.8	227.9	178.7	109.8	10.6	227.4	264.0	277.3	n.a.
Austria	184.4	178.0	157.2	110.6	6.6	173.8	161.2	206.7	192.1
Belgium	202.2	193.6	165.6	110.7	7.8	191.8	n.a.	n.a.	n.a
Canada	202.5	185.9	160 1	112.4	7.6	231.4	160.2	210.0	188.4
Denmark	247.2	224.7	185.5	119.1	9.6	275.6	193.0[3]	n.a.	n.a.
Finland	280 0	260.3	202.0	114.6	11.8	299.4	n.a.	n.a.	n.a.
France	233.9	214.5	178.9	117.1	9.0	246.6	216.2	234.5	257.3
Germany, West	160.6	156.5	144.1	107.0	5.2	150.7	161.9	171.2	167.1
Greece	270.3	240.1	189.0	105.8	12.4	299.4	n.a.	n.a.	n.a.
Iceland	1,200.0[3]	841.8	488.2	159.7	28.7[4]	n.a.	n.a.	n.a.	n.a.
Ireland	327.6	304.2	226.8	121.7	13.2	350.7	n.a.	n.a.	n.a.
Italy	289 8	258.5	186.9	109.2	13.0	292.1	318.1	254.0	334.8
Japan	252.1	241.9	204.5	119.3	9.8	266.2	256.7	204.4	272.8
Luxembourg	188.0	182 4	155.7	109 9	6.9	192.3	n.a.	n.a.	n.a.
Netherlands	212.5	203.8	175.3	115.5	7.9	183.7	231.5	212.0	197.6
New Zealand	284.4	254.0	189.6	116.6	11.8	277.7	n.a.	n.a.	n.a.
Norway	227.2	210.1	176.5	118.0	8.5	230.0	211.7	n.a.	n.a.
Portugal	423.4	371.2	247.6	122.7	16.7	474.6	n.a.	n.a.	n.a.
Spain	350.9	293 2	200.3	113.3	15.2	342.7	n.a.	n.a.	n.a.
Sweden	222.0	201.8	164.3	112.0	8.9	240.2	174.9	236.9	n.a.
Switzerland	164.0	162.3	157.5	108.8	5.3	153.0	156.7	n.a.	n.a.
Turkey	676.8	418 1	282.5	120.1	24.1	n.a.	n.a.	n.a.	n.a.
United Kingdom	316.6	292.4	216.5	117.4	13.2	372.7	255.0	296.2	323.1
United States	195.4	181.5	161.2	116.3	6.7	211.4	159.6	202.8	185.5
Yugoslavia	449.8	391.0	309.2	125.5	17.3	481.5	n.a.	n.a.	n.a.

1. Restaurant meals, alcohol, and tobacco are included for some countries, excluded for others. 2. Includes shelter, utilities, and household furnishings and operations. However, actual coverage and measurement methods vary significantly from country to country. 3. June 1978. 4. 1970 to June 1978. NOTE: n.a. = not available. *Source:* United States Department of Labor, Bureau of Labor Statistics.

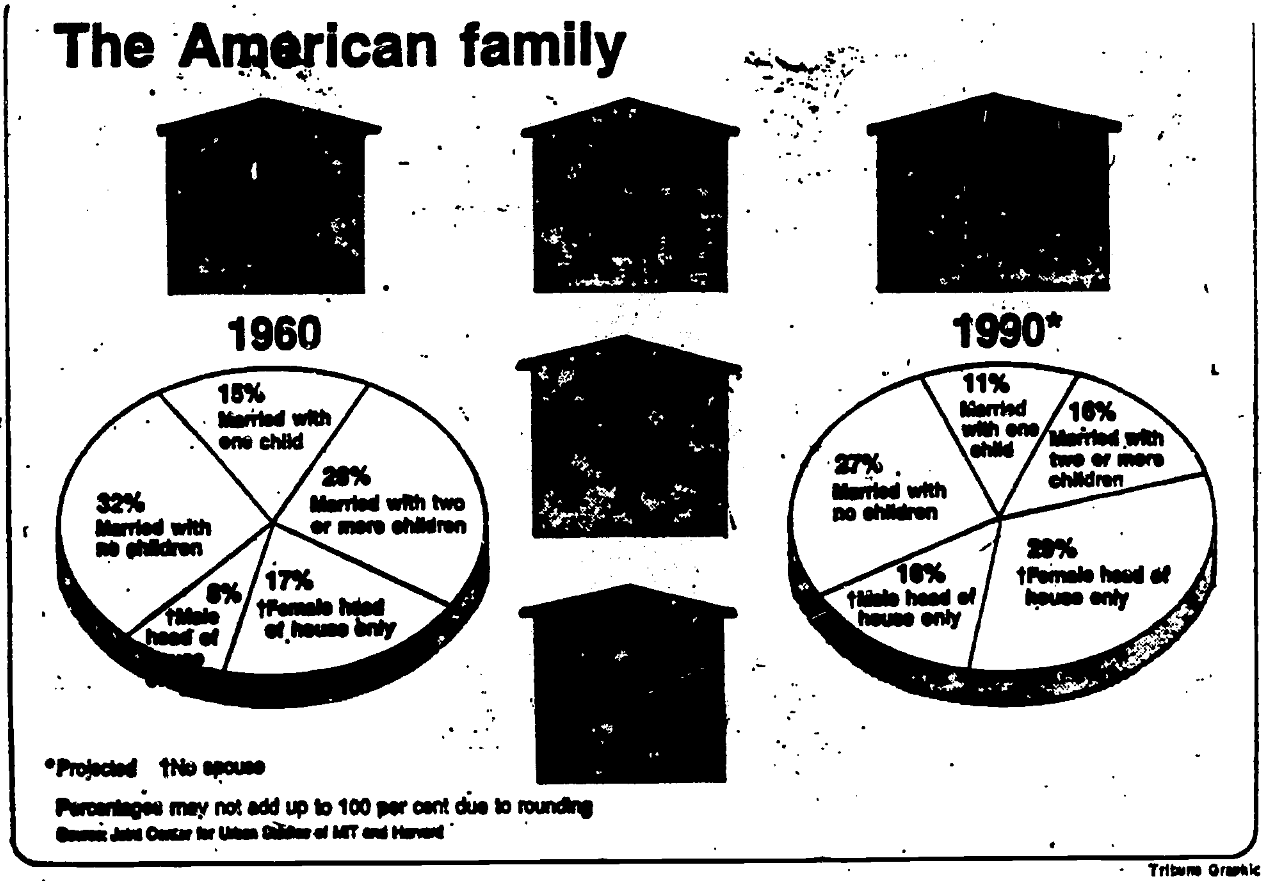
Answer: (a)



(b) The graph shows the large difference that seemingly small differences in average percent yearly inflation can make when carried over a period of 8 years.

Comment: The table contains so much information that a reader has to shut out information in order to see trends. By displaying less information, the trends are more apparent. Notice that though the U.S. inflation was not low in this era, it was one of the lowest of all developed countries.

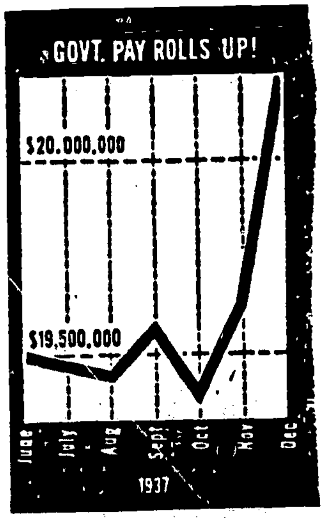
7. Circle graph. The two circle graphs are from a report of a study "The Nation's Families: 1960-1990," sponsored by the Joint Center for Urban Studies at MIT and Harvard, as reported and displayed in the Chicago Tribune. Why were circle graphs chosen to display the given information?



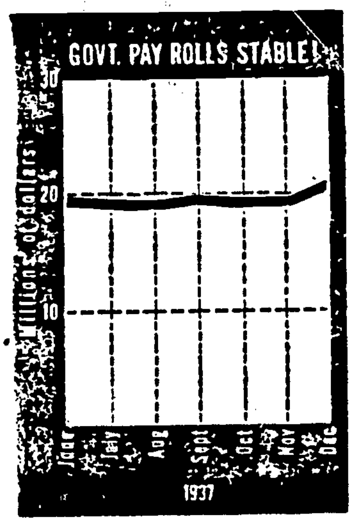
Answer: Trends in families are most reasonably expressed by percentages of families with certain characteristics. These percentages can be clearly displayed in circle graphs.

Comment: The two circle graphs enable easy comparison of 1960 with the projections for 1990. On the other hand, the five shaded houses (one for each category in the pie) distracted us and added no information.

8. Redo the graph below with a vertical scale that goes from 0 to 30 million dollars.

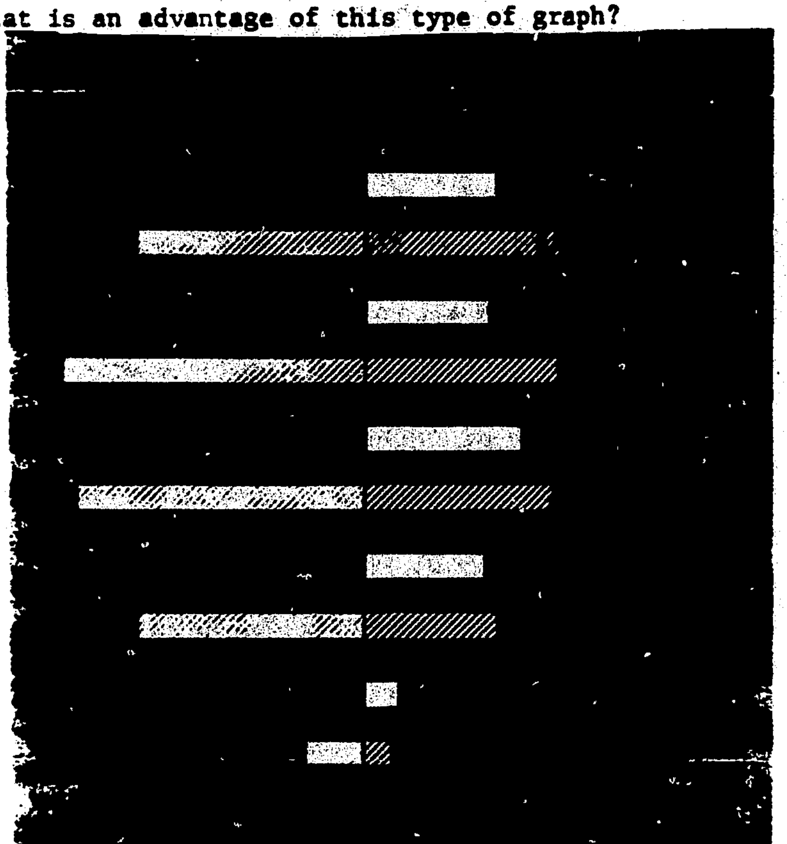


Answer:



Comment: This example is taken from Darrell Huff's famous book, <u>How to Lie with Statistics</u>, p. 65. Neither graph is wrong or a lie, but they convey different impressions (as the titles indicate). The original graph does not start at 0, but that is not a rule that must be followed. Space does not always allow a graph to begin at zero and include all data. In fact, the answer's graph has such a large scale that it could be considered deceptive because it disguises all trends. Clarity is a function of what information you wish to emphasize.

9. Dual-direction bar graphs. The bar graphs given below represent a relatively new trend in graphs, namely the two-direction bar graph. What is an advantage of this type of graph?



Answer: Comparative information (in this case, men and women) can be displayed side by side. An imbalance to one side can easily be seen. In this case, as is to be expected, more men than women work at each age level.

Comment: The Conference Board in July, 1978, stated: "As to labor-force participation, the latest Bureau of Labor Statistics projections show a modest further rise during the next decade in the participation of women aged 45-54 but declining rates for others in the 45-and-over population." Notice how easily trends can be seen.

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Displaying Use Class C: Facility

One often chooses one display over another to make a particular data analysis task easier. An appropriately designed graphical display enables the viewer to estimate missing data and interpolate or extrapolate from given information to an extent that the numbers themselves might not suggest. Prose presentations may include non-numerical information that can help one to work with data. Tables, because they organize so much information, facilitate analysis and comparison.

Examples:

1. What three comparisons can be easily made because of the way that the top part of the following table is organized? (Source: The New York Times, July 19, 1981.)

Survey of High School Students' Use Of Drugs and Alcohol

	Have you ever used		Have most of your friends used		
	YES	NO	YES	NO	NOT SURE
MARIJUANA	54	46	74.2	10.8	14.9
COCAINE	15.2	84.8	29.5	45.1	25.3
PILLS*	19.6	80.4	32.2	43.1	24.7
P.C.P.†	3.7	96.3	7.3	62.8	29.7
QUAALUDES	7.8	92.2	20	51.9	28
HEROIN	0.6	99.4	3.5	67.8	28.6

*Pills not prescribed for respondent
†Phencyclidine, commonly known as Angel Dust

How often do you drink?	CITY	SUBURBS	TOTAL
EVERY DAY	3.6	5.2	4.1
ONCE A WEEK	17.1	25.2	19.9
OCCASIONALLY	52.9	52.9	52.9
NEVER	23.3	14.2	20.2

Source: The New York Times Metropolitan Area High School Survey

Answer: One can compare those who have used a particular drug with those who have not, one can compare the drugs against each other, and one can compare the actual use with the perceived use.

Comment: All of the numbers in this table are percents. Judging from the number of decimal places in them, it seems that the authors dropped zeros even though they would signify greater accuracy. It is probable that the percent that had used marijuana was 54.0.

2. From the table below, calculate the total cost to the buyer of

(a) 4 A78-13 tires and (b) 2 L78-15 tires. (Source: Chicago Sun-Times, July 19, 1981.)

Mileagemaker II

Sale 4 for $144

Reg. $42+1.58 F.E.T. Size A78-13. Our finest four ply tire has a polyester cord body. Whitewalls.

Size	Reg.	Sale	F.E.T
A78-13	42.00	36.00	1.58
B78-13	49.00	42.00	1.71
E78-14	54.00	46.00	2.04
F78-14	55.00	47.00	2.14
G78-14	58.00	49.00	2.28
G78-15	59.00	50.00	2.36
H78-15	62.00	53.00	2.57
L78-15	64.00	54.00	2.84

Answers: (a) $144 + 4($1.58) = $150.32; (b) 2(54.00 + 2.84) = 113.68.

Comment: The headline is misleading, as the sale price does not include Federal Excise Tax. However, separating tax from price in the table makes it easier to calculate the pre-tax savings. (No one is allowed to discount the F.E.T.)

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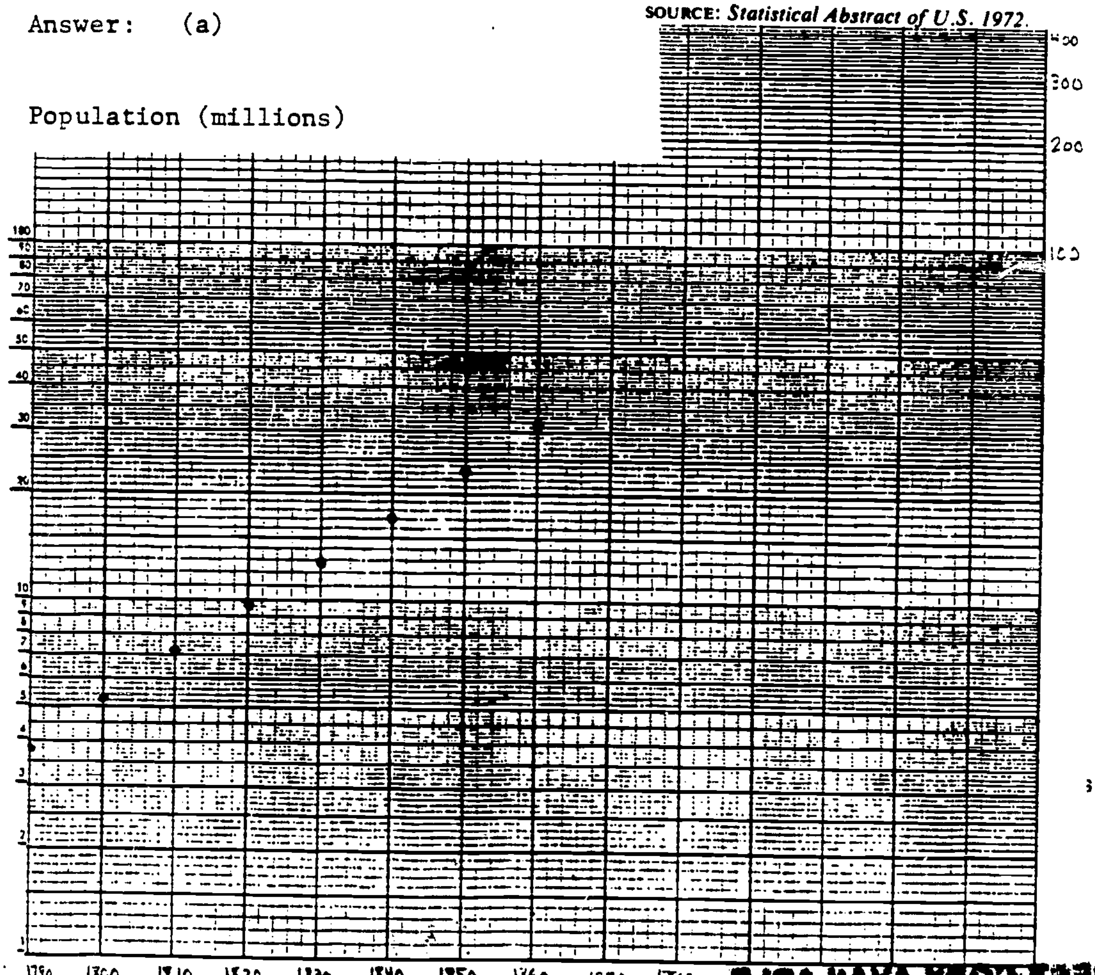
3. Here are some U.S. ce~sus data. It is said to be the case that on the basis of the data up to 1860, President Lincoln estimated that the population by 1930 would be over 250 million. (a) Graph just the data from 1790 to 1860 on semi-log graph paper. (b) Would you have made the same prediction? (c) Graph the data from 1900 to 1970. (d) Use this data to predict the U.S. population in 2010.

Census Year	Population	Census Year	Population
1790	3,929,214	1870	39,818,449
1800.	5,308,483	1880	50,155,783
1810	7,239,881	1890	62,947,714
1820	9,638,453	1900	75,994,575
1830	12,866,020	1910	91,972,266
1840	17,069,453	1920	105,710,620
1850	23,191,876	1930	122,775,046
1860	31,444,321	1940	131,669,275
		1950	151,325,798[1]
		1960	179,323,175[1]
		1970	203,211,926[1]

SOURCE: *Statistical Abstract of U.S. 1972.*

Answer: (a)

Population (millions)



(b) Drawing a line through the 1790 and 1860 populations, one arrives at an estimate of 250 million for the 1930 population. (c) and (d) An estimate similarly done for 2010 yields 350 million.

Comment: Drawing a line through the 1860 and 1790 populations and using that to predict the 1930 population is equivalent to solving the proportion

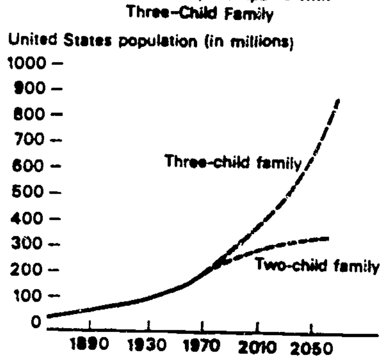
$$\frac{\text{1860 population}}{\text{1790 population}} = \frac{\text{1930 population}}{\text{1860 population}}$$

That is, in millions $\dfrac{31.4}{3.9} = \dfrac{x}{31.4}$

This method yields 252 million for the 1930 estimate and may be what Lincoln did. (It only works because the number of years between 1790 and 1860 equals the number of years between 1860 and 1930.)

Comment: Small changes in the birth rate can lead to large changes in the population. The graph given here shows what a large change in the size of the family can do. The two-child family data comes close to the answer predicted in part (d) of this example.

Anticipated Population Growth in the United States: Two-Child Family Compared with Three-Child Family



4. Use the table below to answer each question regarding the seven countries during 1978–80. (a) Which countries had the greatest and least inflation during this time? (b) Which had the highest and lowest unemployment? (c) Does there seem to be a relationship between inflation and unemployment? (d) What was the total balance of payments for the U.S. in these three years? (e) What trend occurred in British oil imports? (Source: New York Times, July 5, 1981.)

Statistical sketchbook of the big 7

		Britain	Canada	France	Italy	Japan	United States	West Germany
Annual inflation rates (in percent)	1978	8.3	9.0	9.2	12.1	3.8	9.0	4.5
	1979	13.4	9.1	10.8	14.8	3.6	11.3	4.1
	1980	18.0	10.2	13.6	21.2	8.0	13.5	5.5
Annual unemployment rates (in percent)	1978	5.8	8.3	6.1	7.1	[illegible]	6.0	4.3
	1979	6.1	7.4	5.9	7.5	2.1	5.7	3.3
	1980	6.3	7.5	6.3	7.4	2.0	7.0	3.3
Growth of real Gross National Product (percent change from previous year)	1978	3.50	3.40	3.30	2.60	6.00	4.40	3.50
	1979	1.70	2.90	3.20	5.00	5.90	2.30	4.40
	1980	-2.25	0.25	2.00	3.50	5.50	1.00	2.00
Balance of payments (in billions of dollars, seasonally adjusted)	1978	1.20	-4.40	6.70	6.20	16.60	-14.30	8.70
	1979	-3.60	-4.40	1.20	5.10	-8.80	0.80	-5.50
	1980	4.50	-3.50	-7.75	-5.25	-10.25	5.60	-17.25
Oil imports (in millions of tons)	1978	41.7	12.2	105.6	84.2	252.7	402.0	140.6
	1979	19.5	8.2	126.0	99.3	286.0	411.6	145.6
	1980*	-2.3**	9.0	109.0	98.1	243.9	321.2	127.0

*preliminary **excess domestic production Sources: O.E.C.D.; International Energy Agency; French Ministry of Energy

Answers: (a) Italy, West Germany. (b) Canada, Japan. (c) The countries with higher inflation rates tended to have higher unemployment. (d) -8.00 billion dollars, i.e., 8 billion more paid out than taken in. (e) Britain became self-sufficient on oil (due to North Sea drillings).

Comment: Tables make it easy to find, compare, and operate on data, and to spot trends. Many other questions could be posed and easily answered from the information in this table.

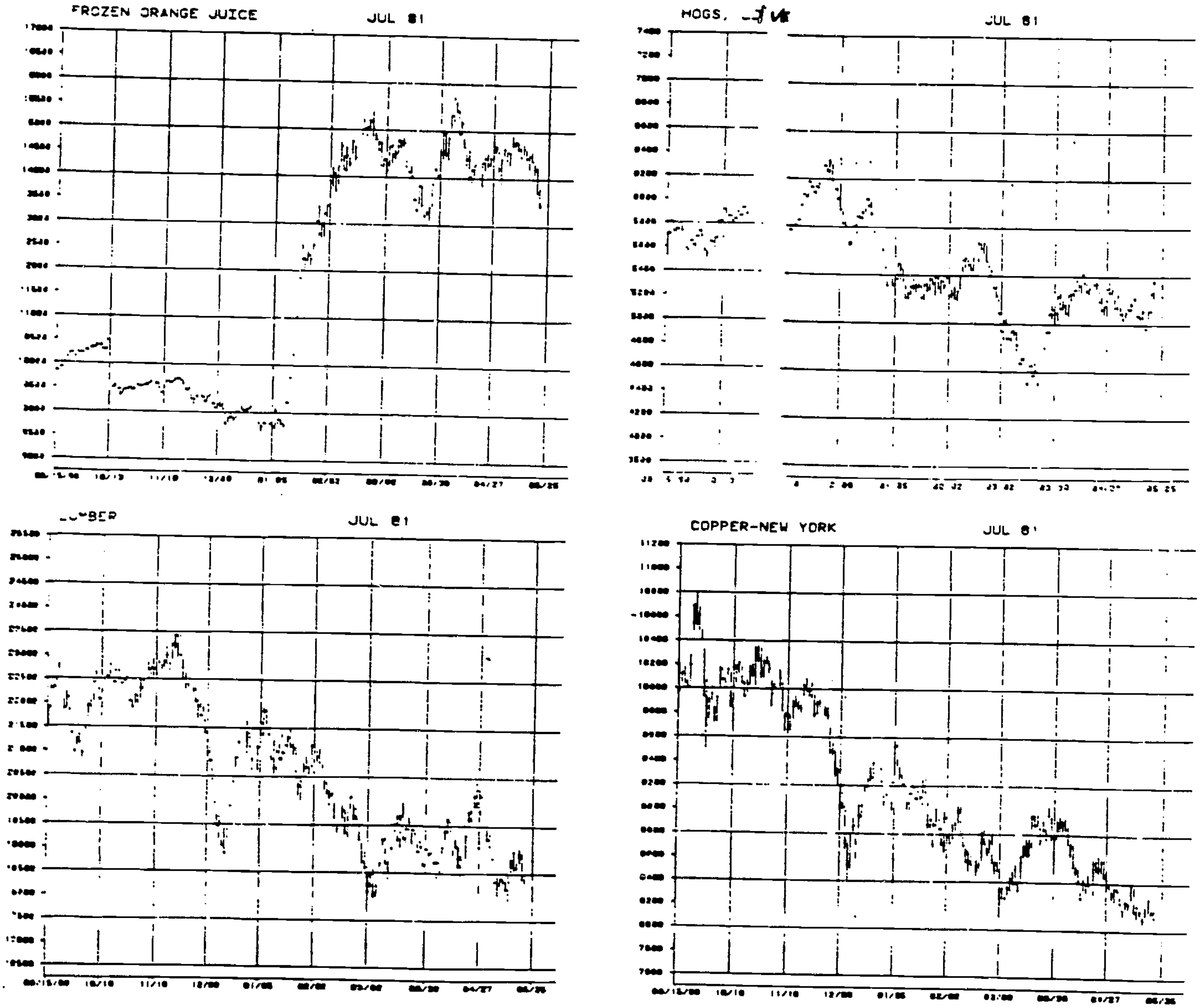
Displaying Use Class D: Consistency

To make sense of numerical data and to search for trends and relationships, the experienced data analyst may rewrite values, make estimates, transform data, or display the data in quite a variety of ways. That is, in exploration of numerical data one often uses all the maneuvers covered in Chapters 11 - 14 of this book, and in particular one might display the same data in different ways. But in communicating results, it is often useful to choose a single kind of display and stick to it throughout a report or in different reports at different times. For example, displays on sports pages or financial pages of newspapers are remarkably consistent from day to day and even from year to year. Such consistency enables a reader to become familiar with abbreviations or codes or standard format used in the displays. The reader can also, when appropriate, more easily make comparisons from one display to the next.

That is, while variety of display is useful in understanding and using numerical information, consistent displays tend to enhance communication of data. Consistency breeds clarity.

Examples:

1. Here are four graphs from an investment newsletter giving information about different commodities. What are the common features of these displays?



Answer: For each graph, the vertical axis indicates the range of prices within which the particular commodity has been traded over the previous 36 weeks. The bar for each day shows the range from minimum to maximum trade price for that day, with a tick mark showing the price at the close of trading. The series of bars shows price trends over the period from September 15, 1980 to about May 20, 1981.

Comment: To interpret the graphs, the reader must be familiar with the field. For example, 5200 on the vertical scale for live hogs means $52.00 was the price per hundred pounds traded.

2. Here are standings, as given daily in many newspapers, for football, basketball, and hockey. (a) What feature do these standings have in common? (b) In each of the three standings, there are one or more columns which contains information that could have been calculated by a reader from information in the other columns. Find these columns. (c) Why would standings contain so much redundant information? (Source: Chicago Tribune, December 1980 and November 7, 1981.)

N.F.L.

Standings

AMERICAN CONFERENCE

Eastern Division

	W	L	T	Pct.	Points For	Points Agt.
Buffalo	10	4	0	.714	[illegible]	[illegible]
New England	8	6	0	.571	379	[illegible]
Baltimore	7	7	0	.500	313	[illegible]
Miami	7	7	0	.500	[illegible]	[illegible]
Jets	3	11	0	.214	[illegible]	[illegible]

Central Division

	W	L	T	Pct.	Points For	Points Agt.
Cleveland	10	4	0	.714	[illegible]	[illegible]
Houston	9	5	0	.643	[illegible]	[illegible]
Pittsburgh	8	6	0	.571	[illegible]	271
Cincinnati	5	9	0	.357	[illegible]	271

Western Division

	W	L	T	Pct.	Points For	Points Agt.
Oakland	9	5	0	.643	[illegible]	[illegible]
San Diego	9	5	0	.643	371	[illegible]
Denver	7	7	0	.500	[illegible]	[illegible]
Kansas City	7	7	0	.500	[illegible]	[illegible]
Seattle	4	10	0	.286	[illegible]	[illegible]

NATIONAL CONFERENCE

Eastern Division

	W	L	T	Pct.	Points For	Points Agt.
*Dallas	11	3	0	.786	[illegible]	[illegible]
*Philadelphia	11	3	0	.786	[illegible]	[illegible]
St. Louis	5	9	0	.357	[illegible]	[illegible]
Giants	4	10	0	.286	219	[illegible]
Washington	4	10	0	.286	214	273

Central Division

	W	L	T	Pct.	Points For	Points Agt.
Minnesota	8	6	0	.571	273	[illegible]
Detroit	7	7	0	.500	[illegible]	[illegible]
Chicago	6	8	0	.429	276	[illegible]
Green Bay	5	8	1	[illegible]	[illegible]	[illegible]
Tampa Bay	5	8	1	[illegible]	[illegible]	[illegible]

Western Division

	W	L	T	Pct.	Points For	Points Agt.
*Atlanta	11	3	0	.786	[illegible]	[illegible]
Los Angeles	9	5	0	.643	[illegible]	[illegible]
San Francisco	[illegible]	[illegible]	0	[illegible]	[illegible]	[illegible]
New Orleans	0	14	0	[illegible]	[illegible]	[illegible]

*Clinched playoff berth.

Scoreboard

Basketball

NBA

Eastern Conference

Central Division

	W	L	Pct.	GB
Detroit	3	1	.750	—
Milwaukee	2	1	.667	½
Indiana	2	2	.500	1
Atlanta	2	2	.500	1
CHICAGO	1	3	.250	2
Cleveland	1	2	.333	2

Atlantic Division

	W	L	Pct.	GB
Philadelphia	4	1	.800	—
Boston	3	1	.750	½
New York	2	2	.500	1½
Washington	1	3	.250	3
New Jersey	0	3	.000	3½

Western Conference

Midwest Division

	W	L	Pct.	GB
San Antonio	3	1	.750	—
Denver	2	1	.667	½
Utah	2	1	.667	½
Kansas City	2	2	.500	1
Houston	2	3	.400	1½
Dallas	1	3	.250	2

Pacific Division

	W	L	Pct.	GB
Portland	4	0	1.000	—
Phoenix	2	2	.500	2
San Diego	1	2	.333	2½
Seattle	1	2	.333	2½
Los Angeles	1	2	.333	2½
Golden State	1	3	.250	3

Hockey

NHL

Campbell Conference

Norris Division

	W	L	T	GF	GA	Pts.
Minnesota	[illegible]	2	2	[illegible]	31	10
CHICAGO	[illegible]	3	[illegible]	[illegible]	[illegible]	15
Detroit	[illegible]	[illegible]	2	[illegible]	[illegible]	14
Winnipeg	[illegible]	3	2	[illegible]	[illegible]	14
St. Louis	4	[illegible]	2	[illegible]	[illegible]	10
Toronto	3	[illegible]	2	[illegible]	[illegible]	[illegible]

Smythe Division

	W	L	T	GF	GA	Pts.
Edmonton	10	4	0	77	[illegible]	20
Vancouver	[illegible]	7	3	[illegible]	47	13
Los Angeles	[illegible]	[illegible]	[illegible]	[illegible]	74	12
Calgary	2	[illegible]	4	44	[illegible]	[illegible]
Colorado	2	[illegible]	2	[illegible]	[illegible]	[illegible]

Wales Conference

Patrick Division

	W	L	T	GF	GA	Pts.
N.Y. Islanders	[illegible]	1	3	[illegible]	[illegible]	19
Philadelphia	[illegible]	4	1	[illegible]	[illegible]	17
Pittsburgh	[illegible]	7	2	[illegible]	[illegible]	14
N.Y. Rangers	[illegible]	[illegible]	[illegible]	44	[illegible]	10
Washington	1	12	0	37	[illegible]	2

Adams Division

	W	L	T	GF	GA	Pts.
Boston	[illegible]	2	3	[illegible]	[illegible]	16
Montreal	7	2	4	73	[illegible]	18
Quebec	[illegible]	[illegible]	[illegible]	71	[illegible]	18
Buffalo	[illegible]	3	4	[illegible]	41	16
Hartford	1	[illegible]	[illegible]	[illegible]	50	7

Answers: (a) They contain the numbers of wins (W) and losses (L), and rank the teams in order within divisions. (b) The Pct. column in football, the Pct. and games behind (GB) columns in basketball, and the Pts. column in hockey are redundant. (c) Except for the GB column, these redundant columns contain the information used to rank the teams. Thus they are, in their own way, more important than the raw data.

Comment: In football, Pct. $= \dfrac{W + .5T}{\text{no. of games played}}$. In basketball, Pct. $= \dfrac{W}{\text{no. of games played}}$. In hockey, Pts. $= 2W + T$.

In all sports,

$$GB = [(W\text{-}L \text{ for 1st place team}) - (W\text{-}L \text{ for given team})] \div 2.$$

(This formula involves a nice application of negative numbers.)

Comment: In the basketball standings, there is an error in the GB column for Cleveland and in the Pct. column for Chicago.

3. Here is a fragment from a display of results for trading on the New York Stock Exchange. This display packs in a great amount of information in a consistent manner known to stock watchers. (a) What does each column stand for? (b) Is any of the information redundant (as in the sports standings of the previous example)?

52-week range High	Low			Yield Pct.	P-E Ratio	Sales (hds)	Week's High	Low	Close	Net Chg.
21	8	RecnEq		—	22	819	11 1/8	11	11 1/8 −	1/8
14 1/4	7 1/2	Redmn	.30	2.4	13	1318	12 3/4	11 1/2	12 3/8 +	5/8
10 1/8	7 1/2	Reece	.60	6.9	13	58	8 3/4	8 1/2	8 3/4 +	1/8
41 1/2	29 1/4	ReevsB	2.20	5.8	5	69	38 1/2	38	38 −	1/2
16	11	ReichCh	.48	3.6	7	180	13 1/4	13 1/8	13 1/4 +	1/8
100 1/4	62 1/2	RelGp	3	3.3	7	1492	u100 1/4	89 1/2	89 3/4 −	8 1/4
150	120 1/2	RelG pf	2.20	1.6	—	11	u150	136	136 −	3 1/2
23 1/2	17 3/4	RelG pf	2.60	14.1	—	16	19	18 1/2	18 1/2 −	1/2
23 1/8	19 1/8	Reiln pf	2.60	13.0	—	67	20 1/8	20 1/8	20 1/8 +	1/8
11 5/8	5 1/4	RepAir	.10	1.1	—	575	8 7/8	8 1/2	8 3/4 +	1/8
35 3/4	19	RepCp	.60	2.0	8	123	31 1/2	30	30 −	1/2
30	16 3/4	RepFnS	1.20	4.9	8	243	24 3/4	24 1/4	24 1/4 −	1/4
4	2 1/2	RepMtg		—	11	55	3 1/8	3	3	
[illegible]	25	RepNY	1.20	3.1	6	203	39 1/4	38 3/4	38 7/8 −	1/8
	15	RNY pf	2.12	12.9	—	8	16 1/2	16 1/4	16 1/2 +	1/2
		RNY pfA	3.13	14.5	—	83	21 5/8	21 1/4	21 5/8 +	3/8
		[illegible]	2	7.3	7	490	27 1/4	25 3/4	27 1/4 +	1/2
			1.40	3.4	7	870	41 5/8	40 3/4	41 5/8 +	5/8
			[illegible]	6.4	—	65	31 1/2	29 3/4	31 1/2 +	1
				2.4	8	539	13 3/4	13 1/2	13 1/2	
					11	497	35 3/4	34 3/4	35 1/2 −	1/4
						2211	24	21 1/2	21 5/8 −	1 1/2
							42	40 1/2	40 1/2 −	1/4
							18 1/4	17 3/4	17 3/4 −	3/8
								16 1/4	17 1/4 +	7/8
									47 3/4 +	1 1/4
									+	5 1/2
										3/8

Answers: (a) We interpret the columns for the stock Rep Air (Republic Airlines). In the last 52 weeks before this day, the stock sold for as high as 11 5/8 dollars per share and as low as 5 1/4 dollars per share. It pays a $.10 yearly dividend, (the yield), which is 1.1% (the Pct.) of its present price. The ratio of the stock price to the past year's company earnings (P-E ratio) is not given (probably because the earnings were negative; i.e., there was a loss). 57,500 shares of stock were traded this day, selling for between 8 1/2 and 8 7/8 dollars per share. The last trade was at 8 3/4 dollars a share, which was 1/8 dollar higher than the last trade on the previous day. (b) The column with the 1.1 (which is actually Pct., but the label is too far left) is redundant.

Comment: 11 5/8 dollars = \$11.625. Odd pennies, when they occur, are rounded.

4. Below is the standard display for showing the cards held by the four players at the beginning of a bridge hand. Who holds the 3 of clubs? (Source: Chicago Tribune.)

Neither vulnerable. East deals.

NORTH
♠ A 7 2
♡ K 10 3
◇ Q J 7 6
♣ Q 9 5

WEST
♠ J 8
♡ A
◇ K 9 5 2
♣ A J 10 7 6 4

EAST
♠ Q 10 5 4
♡ J 8 4
◇ 8 4 3
♣ K 8 3

SOUTH
♠ K 9 6 3
♡ Q 9 7 6 5 2
◇ A 10
♣ 2

The bidding:

East	South	West	North
Pass	Pass	1 ♣	Pass
1 ♠	2 ♡	3 ♣	3 ♡
Pass	4 ♡	Pass	Pass
Pass			

Opening lead: Jack of ♠.

Answer: East. The 3-club sign in the bidding stands for something entirely different.

Comment: The number of codes in this display makes it hard for someone who has not played bridge to understand what is meant. Almost every field or hobby has displays that are easily understood by those in the know and difficult for others. This display is consistent with the seating at a standard bridge table.

Summary

A <u>display</u> of numerical information is a visual presentation of that information. Displays include lists, tables, charts, diagrams, graphs, maps, and prose. The major reasons for selecting one kind of display over another are constraints, or a desire for clarity, facility, or consistency.

A wide range of numbers may constrain one to pick one sort of display over another. Other constraints are the graphics and script capabilities of available hardware. Regarding clarity, a well-constructed display can illuminate relationships in data. Prose and diagrams enable qualitative aspects to be emphasized. Tables and graphs allow a large amount of information to be conveyed without confusion. Combining different kinds of displays can achieve all of these purposes at one time. Displays may facilitate extrapolation, detailed analyses and comparisons with given information, and signal what information is worth using. Communication is enhanced by consistency in display format.

These uses result in a vast assortment of kinds of displays. The displays exhibited in this chapter constitute only a sample of the possibilities.

Pedagogical Remarks

Since numerical information found in books and periodical literature is displayed in some format, there is never a problem finding displays. In fact, a large newspaper or magazines like Time or Newsweek may have more graphs, charts, and tables than a textbook that discusses the topic. Our problem in writing this chapter was to select from the myriads of types of displays, not to find them.

Basic skills. The first skill in work with displays is reading. Wallis and Roberts, in The Nature of Statistics (a very fine book, and quite easy to read), remind us that

> "Ordinary reading ability is no more effective in reading a table than an ordinary can opener in opening a can of sardines, and if you go at it with a hammer and chisel you are likely to mutilate the contents." (p. 195)

That is, reading tables (and the same goes for graphs and charts) requires careful attention and different skills than ordinary reading.

To read a display, it helps to know the components of the particular type of display being read. A graph has two axes upon which are scales. A bar graph has only one scale. A table has headings. Any display may have descriptive information in prose.

The second skill in work with displays is interpretation. What value is largest? Which smallest? Are there any patterns in the data? What is the purpose of the display? Is anything is the display misleading? Is anything surprising? With real data, there are an unlimited number of such questions. The idea in teaching students to interpret displays is to increase their sensitivity to the various aspects of displays: large sectors or small sectors in circle graphs; bars of about the same length in bar graphs; high points or low points in coordinate graphs; subjective information in prose; trends in tables.

A third skill is the construction of displays and the conversion from one kind of display to another. Such work takes time even for adults, and the teacher should be patient with the student who may take 40 minutes to construct what seems to be a simple graph or chart.

At what grade level are students ready to deal with graphs, charts, and tables? The beliefs of teachers are surprisingly varied. We have seen second grade classrooms in which children are _producing_ graphs and we have seen eighth grade classrooms in which teachers feel that _reading_ graphs is beyond their students. We prefer to be optimistic.

Clarity. Students tend not to realize the importance of carefully constructed scales. This must be emphasized. A display takes a long time to construct and that time is rather wasted if the information on the display cannot be easily understood.

If students are constructing their own charts, tables, or graphs, you may wish to allow a great deal of flexibility so that students can decide for themselves what is clear and what isn't. Clarity is often a matter of taste.

Facility. Pick a display out of the newspaper that is particularly rich in data and ask about all of the things that can be determined from the display. For example, a display of prices may give original and sales prices but not the percentage of discount or the amount saved. Milk the display for all that can be gotten from it.

Consistency. Have students bring in displays from hobbies or specialized pursuits. These almost always have special languages (as in the bridge example or the stock market prices). Help the students to explain these special languages to each other. The more varied an assortment of displays a student sees, the less fearful the student will be when a novel kind of display presents itself.

A good display to work with at the beginning of the school year is one of high and low temperatures in your area. Affix each day's temperature. In the northern part of the country, this display will then extend naturally into a discussion of negative numbers (which come earlier for Celsius than for Fahrenheit temperatures). We know of cases where this has been successful with first graders.

Constraints. We do not recommend working on ratio scales at the elementary level. There are more important things to do. But headings like "in thousands" at the top of a column of numbers in order to save space are often used and can be fit in with lessons on computation with powers of 10.

Questions

1. Examine the table. (a) About how many pieces of numerical information are in it? (b) In what countries does a 60-year-old male have the lowest average future lifetime? (c) Why might this data not be accurate today? (Source: Information Please Almanac 1981, p. 130.)

Expectation of Life by Age and Sex for Selected Countries

		Average future lifetime in years at stated age											
		Males						Females					
Country	Period	0	1	10	20	40	60	0	1	10	20	40	60
NORTH AMERICA													
United States	1975	68.7	68.9	60.3	50.8	32.6	16.8	76.5	76.6	67.9	58.1	39.0	21.8
Canada	1970–72	69.3	69.8	61.2	51.7	33.2	17.0	76.4	76.6	67.9	58.2	39.0	21.4
Mexico	1975	62.8	68.0	59.2	49.9	32.7	17.9	66.6	69.3	62.6	53.1	35.3	19.1
Puerto Rico	1976	70.2	70.8	62.1	52.3	34.8	19.0	77.1	77.5	68.7	58.9	39.7	22.0
Trinidad and Tobago	1970	64.1	65.6	57.3	47.8	29.5	13.6	68.1	69.3	60.9	51.3	32.7	16.3
CENTRAL AND SOUTH AMERICA													
Brazil	1960–70	57.6	—	56.2	47.0	30.0	15.0	61.1	—	58.9	49.7	32.5	16.6
Chile	1969–70	60.5	64.9	56.7	47.3	29.9	15.5	66.0	70.0	62.2	52.7	34.5	18.0
Colombia	1950–52	44.2	50.4	48.2	39.6	24.8	11.8	45.9	51.1	49.4	40.9	26.6	12.8
Costa Rica	1972–74	66.3	69.1	61.3	51.8	33.8	17.4	70.5	72.7	64.9	55.2	36.5	19.2
Ecuador	1962–74	54.9	60.5	55.5	47.1	31.3	16.2	58.1	62.8	58.0	49.3	33.0	17.3
Guatemala	1963–65	48.3	52.5	51.3	43.2	28.1	14.8	49.7	53.4	52.8	44.6	29.2	14.7
Panama	1970	64.3	66.4	59.9	50.8	33.4	17.0	67.5	69.4	62.7	53.6	36.1	19.9
Uruguay	1963–64	65.5	68.0	59.5	50.0	31.7	15.9	71.6	73.7	65.2	55.5	36.7	19.5
Venezuela[1]	1961	66.4	68.8	61.8	52.4	34.6	18.9	—	—	—	—	—	—
EUROPE													
Austria	1976	68.1	—	59.9	50.4	32.0	15.7	75.1	—	66.6	56.8	37.6	19.7
Belgium	1968–72	67.8	68.4	59.9	50.3	31.6	15.2	74.2	74.5	65.9	56.1	36.9	19.2
Cyprus	1973	70.0	70.7	62.1	52.3	33.5	16.5	72.9	74.2	65.7	55.8	36.4	18.5
Czechoslovakia	1970	66.2	67.0	58.4	49.9	30.6	14.6	72.9	72.6	64.9	55.2	35.9	18.3
Denmark[2]	1975–76	71.1	70.9	62.3	52.6	33.7	17.1	76.8	76.5	67.7	57.9	38.5	21.1
Finland	1975	67.4	67.1	58.5	48.9	30.6	15.0	75.9	75.6	66.8	57.0	37.7	19.7
France	1976	69.2	69.2	60.5	51.0	32.5	16.7	77.2	77.0	68.3	58.6	39.3	21.5
Germany, East[3]	1976	68.8	69.0	60.4	50.8	32.2	15.9	74.4	74.3	65.6	55.9	36.6	18.8
Germany, West[3]	1975–77	68.6	69.0	60.4	50.9	32.3	15.8	75.2	75.4	66.7	56.9	37.7	19.9
Greece	1970	70.1	72.2	63.8	54.1	35.1	17.5	73.6	75.3	66.9	57.1	37.8	19.3
Hungary	1974	66.5	68.2	59.5	49.9	31.5	15.5	72.4	73.7	65.1	55.3	36.1	18.7
Ireland	1970–72	68.8	69.2	60.6	51.0	32.1	15.6	73.5	73.8	65.1	55.3	40.8	18.7
Italy	1970–72	69.0	70.1	61.6	52.0	33.2	16.7	74.9	75.8	67.1	57.3	38.1	20.2
Netherlands	1977	72.0	71.8	63.2	53.5	34.4	17.4	78.4	78.1	69.4	59.6	40.2	22.1
Norway	1976–77	72.1	71.9	63.2	53.6	34.7	17.7	78.4	78.1	69.4	59.5	40.0	21.8
Poland	1970–72	66.8	68.0	59.4	49.8	31.6	15.5	73.8	74.6	66.0	56.2	37.0	19.3
Portugal	1974	65.3	67.2	59.0	49.6	31.5	15.4	72.0	73.5	65.3	55.6	36.5	18.7
Spain	1970	69.7	—	62.0	52.4	33.7	17.0	75.0	—	66.9	57.1	38.0	20.1
Sweden	1972–76	72.1	71.9	63.1	53.5	34.7	17.6	77.8	77.4	68.6	58.8	39.5	21.4
Switzerland	1968–73	70.3	70.5	62.0	52.4	33.6	16.7	76.2	76.2	67.6	57.8	38.4	20.4
U.S.S.R.	1971–72	64.0	—	—	—	—	—	74.0	—	—	—	—	—
United Kingdom													
England and Wales	1974–76	69.6	—	—	—	—	—	75.8	—	—	—	—	—
Northern Ireland	1975–77	67.5	67.9	59.3	49.7	31.2	14.9	73.8	74.2	65.5	55.7	36.4	19.0
Scotland	1971–73	67.2	67.8	59.2	49.8	31.3	15.0	73.6	74.1	65.4	55.6	36.3	18.9
Yugoslavia	1970–72	65.4	68.0	59.7	50.1	31.8	15.7	70.2	72.8	64.6	54.9	35.8	18.2
ASIA													
Bangladesh	1974	45.8	53.5	50.5	42.5	28.1	14.4	46.6	53.5	50.3	42.2	27.7	14.1
India	1951–60	41.9	48.4	45.2	36.9	22.1	11.8	40.6	46.0	43.8	35.6	22.4	12.9
Iran	1973–76	57.6	63.5	59.1	49.7	31.4	16.1	57.4	64.0	60.9	51.6	33.7	17.9
Israel	1977	71.3	71.7	63.0	53.5	34.6	17.6	74.7	74.9	66.3	56.5	37.1	19.2
Japan[4]	1974	71.2	71.0	62.5	52.8	33.0	17.0	76.3	76.0	67.4	57.5	38.3	20.3
Korea, South	1970	63.0	66.0	58.0	49.0	31.0	16.0	67.0	69.0	61.0	52.0	34.0	17.0
Pakistan	1962	53.7	60.6	57.0	47.8	30.8	15.6	48.8	53.9	51.7	42.9	27.7	15.5
Sri Lanka	1967	64.8	67.4	60.5	51.2	33.2	17.0	66.9	68.9	62.4	53.0	35.0	17.6
Syria	1970	54.5	60.7	56.4	47.4	30.5	15.2	58.7	64.1	59.5	50.5	33.3	17.3
AFRICA													
Egypt	1960	51.6	56.2	56.6	47.7	30.5	15.1	53.8	59.9	62.0	52.9	35.0	18.0
Kenya	1969	46.9	52.6	51.0	43.0	28.3	14.5	51.2	56.6	54.1	45.7	30.3	15.7
South Africa													
(white population)	1959–61	64.7	65.9	57.5	48.0	30.2	15.0	71.7	72.5	64.1	54.4	35.5	18.6
OCEANIA													
Australia[5]	1965–67	67.6	68.1	59.5	50.0	31.4	15.8	74.2	74.4	65.8	56.0	36.9	19.5
New Zealand	1970–72	68.5	—	—	—	—	—	74.6	—	—	—	—	—

1. Figures for male and female together. 2. Excluding data for Faeroe Islands and Greenland. 3. Includes relevant data relating to Berlin. No separate data have been supplied. 4. Japanese nationals in Japan only. 5. Excludes full-blooded aborigines. *Source:* United Nations *Demographic Yearbook, 1978.*

2. (a) According to the ad, what were the sales of the New England Rare Coin Galleries in 1980 and what were the sales of their nearest competitor? (b) What feature of the display is rather deceptive?

3. Graph the data for the three sub-categories for 1940 through 1979 on the same graph. Estimate from your graph what the indexes will be in 1985.

Consumer Price Indexes
(1967 = 100)

Year	Commodities	Services	Housing	All items	Percent change[1]	Year	Commodities	Services	Housing	All items	Percent change[1]
1940	40.6	43.6	52.4	42.0	1.0	1965	95.7	92.2	94.9	94.5	1.7
1945	56.3	48.2	59.1	53.9	2.3	1970	113.5	121.6	118.9	116.3	5.9
1950	78.8	58.7	72.8	72.1	1.0	1975	158.4	166.6	166.8	161.2	9.1
1955	85.1	70.9	82.3	80.2	−0.4	1978	187.1	210.9	202.8	195.4	7.7
1960	91.5	83.5	90.2	88.7	1.6	1979	208.4	234.2	227.6	217.4	11.3

1. Over previous year. Source: Department of Labor, Bureau of Labor Statistics.

4. Consider the most recent day in which you worked for pay. Graph the amount of time spent (a) working, (b) eating, (c) sleeping, (d) in transit, and (e) other. What kind of graph is most appropriate?

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5. Reorder the data below by year. What information does the reordering make more evident? (Source: Irving Wallace et. al., The Book of Lists #2, p. 199.)

THE 25 ALL-TIME BOX-OFFICE CHAMPION FILMS (Adjusted for Inflation)

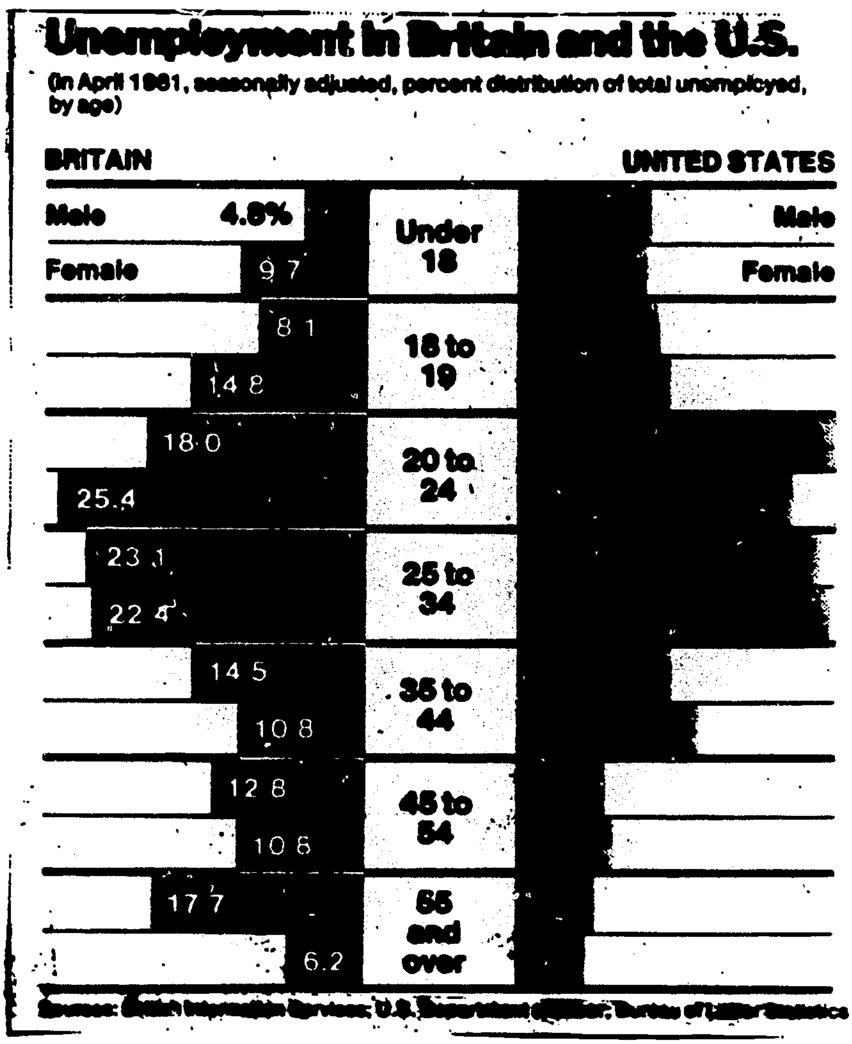
Usually when the top money-making films are tabulated, the ravages of inflation over the years are ignored. For example, on a strict dollar basis, *Star Wars* (1977) has already reaped $100 million more than *Gone With the Wind* (1939). But this doesn't take into account the fact that the 1939 dollar was worth nearly five times as much as the 1977 dollar, and the average price of a theater ticket in 1977 was $2.23 as compared to 23¢ 38 years earlier. Here, then, is a ranking of the top box-office champs, compiled by Charles Schreger, incorporating the inflation factor.

	U.S.-Canada Rentals in Millions of Dollars
1. *Gone With the Wind* (1939)	382.7
2. *Star Wars* (1977)	187.8
3. *The Sound of Music* (1965)	173.8
4. *Jaws* (1975)	156.4
5. *The Godfather* (1972)	143.2
6. *Snow White* (1937)	128.9
7. *The Exorcist* (1973)	128.2
8. *The Sting* (1973)	123.1
9. *The Ten Commandments* (1956)	109.7
10. *Doctor Zhivago* (1965)	102.4
11. *The Graduate* (1968)	97.7
12. *Mary Poppins* (1964)	91.4
13. *Love Story* (1970)	89.0
14. *Grease* (1978)	88.1
15. *Close Encounters of the Third Kind* (1977)	87.8
16. *Ben-Hur* (1959)	87.2
17. *American Graffiti* (1973)	87.2
18. *Butch Cassidy and the Sundance Kid* (1969)	87.0
19. *Saturday Night Fever* (1977)	81.5
20. *Airport* (1970)	80.6
21. *One Flew Over the Cuckoo's Nest* (1975)	76.1
22. *The Poseidon Adventure* (1972)	69.7
23. *Rocky* (1976)	65.9
24. *M*A*S*H* (1970)	65.4
25. *Smokey and the Bandit* (1977)	65.3

6. From any source you can find, locate a graph constructed with a ratio scale and indicate why it seems that a ratio scale was used.

7. For which sexes and which age groups was there most difference in unemployment between Britain and the U.S. in April, 1981?

(Source: The New York Times, July 19, 1981.)



8. (a) Transform the data given in the table to the circle graph.

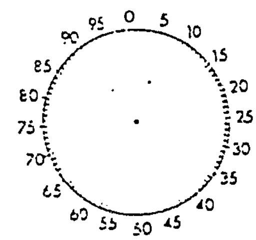
   (b) What does the circle graph clarify?

Annual Budget for 4-Person Family (1979)

Food	$5000
Housing	4600
Transportation	1800
Medical	1200
Taxes*	4200
Other	3000
Total	$19,800

*Including social security

Where the Money Goes



9. In designing calendars, the situational constraint is that one must take numbers from 1 to 31 (or 30 or 29 or 28), arrange them among days of the week, and come up with a consistent design for all the months of a given year or years. Locate at least one unusual calendar design and indicate its advantages or disadvantages compared to the typical rectangular one-month array.

10. Baseball standings typically include numbers of wins and losses, the percentages of games won, and games behind for all teams but the leaders. (a) Which of this information is redundant? (b) Why is redundant information included?

11. Find an interesting display different in some way from any mentioned in this chapter. Indicate what it is that makes the display so interesting to you.

Notes and Commentary

1. Non-arithmetic displays
2. Related work
3. Other possible classifications of uses of displays

1. Non-arithmetic displays. There are at least four common kinds of displays that were purposely left out of the discussion in this chapter: flow charts, displays of functions, displays which seemed to us to be geometric than arithmetic, and displays in which a standard format is highly embellished by graphics tied to the situational context. Here are examples of all of these kinds but flow charts.

Displays of functions

(Source: F. Stine, Handbook of Model Rocketry, G. Follett, 1967, p. 56.)

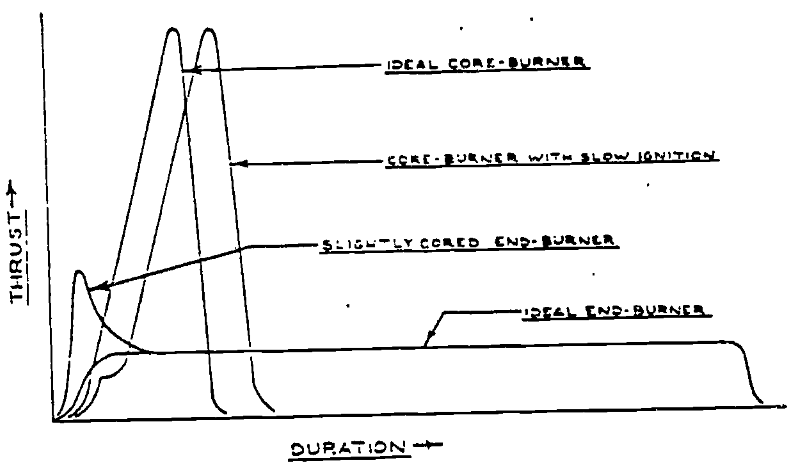
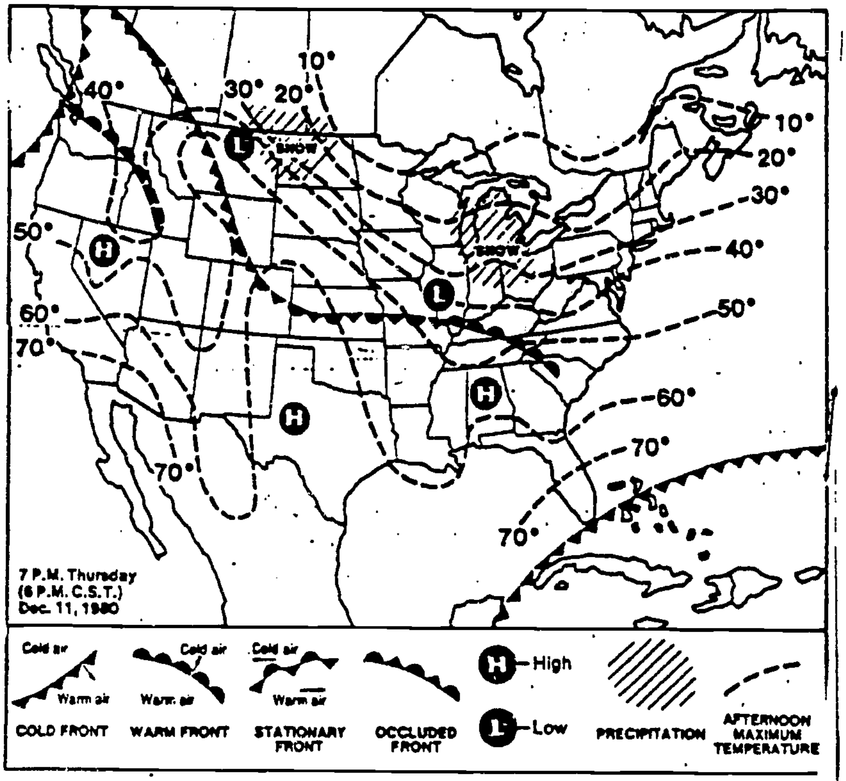


Figure 3-7: Thrust-time curves of various types of model-rocket engines.

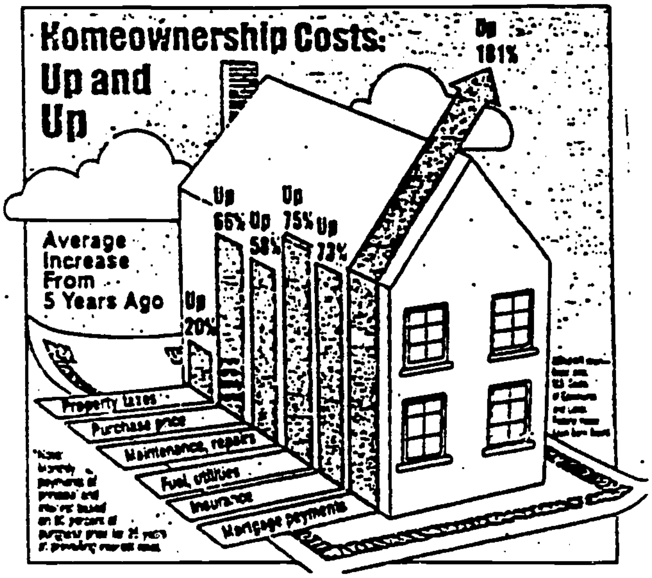
Geometric displays (Source: Chicago Tribune, March 15, 1981)

National Forecast

Snow showers are likely to fall across the Ohio Valley, the lake region, the Northern Plains States and the upper Mississippi Valley. Freezing rain will probably fall in Northern New England and the Pacific Northwest. Temperatures will be warmer across the Northern and Central Rockies and skies will be clear elsewhere.



Embellished arithmetic displays (Source: U.S. News and World Reports, July 27, 1981)



Our rationale for restricting the kinds of displays was twofold: space considerations and a desire to restrict discussion to those displays that were clearly tied to numerical information.

2. Related work. A fine reference on the various types of displays is Graphical Analysis, by Philip Stein. Recently, due primarily to the work of Tukey (1977), "exploratory data analysis" has become a sopisticated look into the techniques of display and tranformation of numerical information. Such analysis includes a large number of interesting ways of displaying numerical information for the purposes of clarity and teasing out relationships. Among these, stem and leaf displays seem most likely to appear in future schoolwork (see Tukey, pp. 8-16).

William Kruskal (1982) has written a superb discussion of criteria for judging displays.

We know of no work that has attempted a classification of displays such as we have given in this chapter.

3. Other possible classifications of uses of displays. As with the other chapters in this part, a natural alternate classification can be achieved by first deciding upon what types of displays are to be considered, and then examining the specific reasons each type of display is used. For instance, tabular displays are used because they enable a great deal of information

to be put into a small space and because they can be typed on a typewriter.

The result of doing this kind of classification would be a more detailed rendering of the concepts of clarity, facility, consistency, and situational constraints than is given here, and would be a welcome contribution. Because of our decision to be consistent within the chapters of this part, we did not undertake such a task.

Summary of Part III

Each of the maneuvers has the same use classes. The use classes interact with the maneuvers in varying ways, as depicted in this table.

Maneuver/ Use Class	Constraints	Clarity	Facility	Consistency
Rewriting	to fit hardware to fit algorithms	for ease of understanding	for ease of computation or comparison	to simplify to conform to stylistic requirements
Estimating	required by measurements, variability, lack of knowledge, prohibitive cost of obtaining exact value, safety margins	for ease of understanding	for ease of computation or comparison	to conform to customary precision, digits, or stylistic requirements
Transforming	to fit space constraints to compress data or fit mathematical constraints	for ease of interpretation	for ease of computation	to equate data from dissimilar contexts
Displaying	to fit hardware to compress data or fit mathematical constraints	to emphasize or illuminate relationships	for ease of extrapolation or comparison	to facilitate communication

Postscript

It was 1979, after each of us had written separately on various aspects of applications, when we decided to write together on the applications of arithmetic. We envisioned a monograph of 100-125 pages, structuring and detailing the uses of numbers and operations. That we would wind up with a manuscript four to five times as long was far from our minds.

The size of the manuscript grew due to a variety of reasons. First was the decision to include in our schema objects other than single numbers. That led to a sizeable Part I. Second was the conceptualization of what we later called maneuvers, giving rise to Part III. Third was the view of our Advisory Boards that, for a book of this kind, examples are as important as generalizations, leading us to include a variety of examples (rather than just two or three) in most sections. Fourth was our own view that the thinking that went into our decisions regarding categorization is as important as our final product, resulting in a collection of notes and commentary.

With the manuscript growing like Topsy, the maintenance of the visibility of the important ideas became a difficult task. We decided on comments after each example, pedagogical comments after each chapter, and further notes and commentary after each chapter so that we could separate out the various kinds of statements we wished to make.

Along the way we wrote essays, sometimes to each other, sometimes for a wider audience, on those ideas that we felt were particularly important and not well-enough treated elsewhere. These essays in briefer form were subsumed into the book. For example:

Differing views of ratio (note 4, Chapter 2)

Classification of uses of operations (notes 2-4, Chapter 5)

Other kinds of meaning than use meanings (note 5, Chapter 5)

Unit arithmetic and dimensional analysis (note 6, Chapter 7)

Repeated addition as not a use of multiplication (Pedagogical Comments, Chapter 7)

Conceptual confusion regarding fractions and ratios (notes 8-11, Chapter 9)

Uses of operations as a postulate set (note 1, Chapter 10)

Necessity to estimate (Section A, Chapter 12)

Semantics dealing with estimation (notes 1 and 3, Chapter 12)

Despite all of these added features, and despite occasional bickering and even arguments between us regarding many aspects of this book, the beliefs that motivated the writing from the beginning are still with us four years later.

1. The reason many people have trouble applying arithmetic is that they have not been taught the application concepts (use meanings) that such applications involve. Because they have not been taught the concepts, they must proceed by feel, and many choose not to proceed at all.

2. There are basic uses for number objects, operations, and maneuvers of arithmetic. These uses are appropriate first ideas in learning how to apply arithmetic.

3. The calculator has already had a profound effect on the ways adults do arithmetic. It should have a similar effect upon the ways children learn arithmetic Specifically, much of

the time now spent doing arithmetic should be replaced by time spent learning when and why that arithmetic is appropriate.

We are disappointed that the calculator remains a forbidden tool in many if not most elementary school classrooms. To us this is akin to forcing children to use a chisel on stone after paper and pencil have been invented. Yet we fear that, when calculators do attain their rightful presence in these classrooms, there will be a tendency to avoid all arithmetic, because teachers will not think there is anything to teach once the skills are taken care of.

We are worried that some people have interpreted our attention to applications of arithmetic as signifying a view that children do not need to learn the mathematical properties underlying number systems, operations, and maneuvers. Let it be clear: the interplay between application and abstraction, and between concrete example and formal generalization, helps the learning of both use and theory. This book is designed to complement corresponding volumes dealing with what is basic to understanding the mathematical properties underlying arithmetic. It does not and should not replace such volumes.

We have been pleased and gratified by the interest shown by colleagues in schools and universities in this work, and by the impact that earlier drafts and conceptualizations have already had on curriculum guides, textbooks, and research.

Zalman Usiskin and Max Bell
June, 1983

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